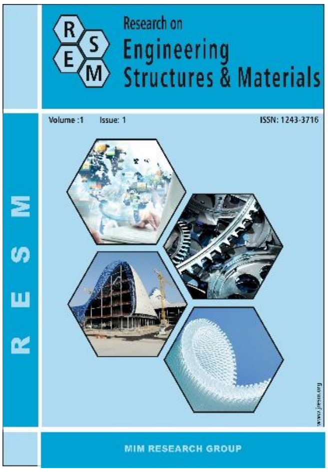
Research on
Engineering
Structures & Materials
journal homepage: http://jresm.org

MIM RESEARCH GROUP



On the mechanics of corbelled domes: new analytical and computational approaches

Pilade Foti, Aguinaldo Fraddosio, Nicola Lepore, Mario Daniele Piccioni

Online Publication Date: 15 Feb 2017
URL: http://dx.doi.org/10.17515/resm2016.71st0725.html
DOI: http://dx.doi.org/10.17515/resm2016.71st0725

Journal Abbreviation: *Res. Eng. Struct. Mat.*

To cite this article

Foti P, Fraddosio A, Lepore N, Piccioni MD. On the mechanics of corbelled domes: new analytical and computational approaches. *Res. Eng. Struct. Mat.*, 2017; 3(3): 210-226.

Research on Engineering Structures & Materials
journal homepage: http://jresm.org

Research Article

On the mechanics of corbelled domes: new analytical and computational approaches

Pilade Foti[1], Aguinaldo Fraddosio[1], Nicola Lepore[1], Mario Daniele Piccioni[1]

Dipartimento di Scienze dell'Ingegneria Civile e dell'Architettura, Politecnico di Bari, Bari, ITALIA

Article Info

Article history:
Received 7 Jul 2016
Revised 1 Feb 2017
Accepted 8 Feb 2017

Keywords:
Corbelled vaults,
Historic constructions,
Equilibrium analysis,
Thrust network analysis,
Corbelling theory

Abstract

The aim of this paper is to present new analytical and computational approaches for assessing the structural safety of "false vaults" structures like *Trulli*, and more generally for corbelled structures.

Starting from a deep investigation on the building techniques of *Trulli* and on the employed materials, we underline that the stability of such structures is justifiable only by admitting the transmission of forces along parallels that is by admitting the three-dimensional nature of their structural behavior.

We proposed two equilibrium methods for assessing the stability of such complex masonry structures, both capable of taking into account their three-dimensional behavior.

The first method is based on the Thrust Network Analysis, a three-dimensional computational method for finding compression only spatial networks in equilibrium with the external loads contained in the thickness of the masonry. The second method is based on a further improvement of the so-called Modified Corbelling Theory, an analytical approach based on the limit overturning equilibrium and specifically developed for the analysis of corbelled domes.

1. Introduction

In the Mediterranean area buildings covered by corbelled vaults made of purely horizontal stone layers, slightly cantilevered toward the center until meeting at the top, comprise a widespread and valuable heritage that deserves protection. In particular, *Trulli* are a remarkable example of vernacular corbelled dry-stone structures, typically located in Apulia, Italy. Whereas this kind of constructions may appear to be the result of a self-building process, actually *Trulli* are the expression of a considerable constructive knowledge, handed down over time. Indeed, the above architectural typology come from the Mycenaean Tholoi built during the late Bronze Age (Greece XIV century B.C.).

The arrangement of *Trulli* vaults in horizontal courses of slightly cantilevered stone elements is obtained by using a very simple technique that requires a minimum level of workmanship on the material and, at same time, do not need the installation of temporary supporting structures, generally employed during the building of ordinary domes.

Because their peculiar construction technique, "false vaults" like *Trulli* structurally behave differently from other kind of masonry vaults; nevertheless, and in spite of their extensive presence, this kind of masonry constructions have received little attention from

*Corresponding author: nicola.lepore@poliba.it
DOI: http://dx.doi.org/10.17515/resm2016.71st0725

the scientific community, and consequently their mechanics is not completely understood yet.

In this paper, starting from an investigation on the building techniques of *Trulli* and on the employed materials, we underline the main mechanical issues of their structural behavior, and we propose two equilibrium approaches within the framework of the Limit Analysis: the first based on the Thrust Network Analysis [4] and the second based on the Corbelling Theory [3].

The first examined approach, that is a generalization of the "thrust line analysis" used for arches, consists in finding a purely compressed membrane in equilibrium with the external loads and entirely contained in the thickness of the vault.

To this aim, an efficient technique – recently proposed in the literature – is represented by the Thrust Network Analysis (TNA), a three-dimensional computational method for finding compression only spatial networks in equilibrium with some external loads, which approximate a continuum compressed membrane.

The integration of the TNA method with optimization algorithms allows for finding possible three-dimensional equilibrium compression spatial networks which are forced to lay within the thickness of a masonry vault.

From a critical analysis of the obtainable results, we found that computational methods like TNA still present several limitations for applications to corbelled vaults, and then they do not rule out other approaches, specifically dedicated to the structural analysis of these structures, and capable of understanding their peculiar behavior.

In this vein, we adopt also a second approach, based on the Corbelling Theory [3]. In particular, we apply a new formulation, which represents a refinement of the Modified Corbelling Theory proposed in [19], which is – at our knowledge – the most effective approach in the literature. The obtained results show that the proposed approach is capable of an accurate description of the actual structural behavior of the examined case study. Moreover, we show that our approach significantly improve that in [19].

2. Building Technique and Structural Behavior of *Trulli*

Trullo is normally a corbelled dry-stone building, covered usually with a pseudo-conical dome. It is generally composed by two main structural elements: the basement and the dome.

The basement, which can be circular or quadrangular, is made up of layers of large calcareous stones; the thickness of the walls is generally large: in most cases the walls are around 1 meter thick.

The dome usually has a pseudo-conical shape, and it is composed of three main parts (Fig. 1). The internal layer, generally with a constant thickness, is known as "*candela*" and it is built by laying subsequent separate jutting rings of stones, without usage of mortar. Special care is dedicated to achieve continuity of the ring brickwork, filling gaps between blocks through little pieces of stones. The finishing of the *Trullo* dome is made by a coating of "*chiancarelle*", thin layers of limestone with a thickness of about *4-8 cm* that are placed on the filling material with an outward slope in order to facilitate the water drainage. Only the first layer *(candela)* has a structural function, and consequently the other two layers can be regarded only as non-structural masses.

The structural behavior of corbelled vaults was addressed in a limited number of studies: for example, since the 1980s some works were focused on the structural analysis of Mycenaean tholoi. In particular, in [7] an in-depth structural analysis on five Mycenaean tholoi is carried out using a multidisciplinary approach, connecting archeological data to the static framework of the so-called Corbelling Theory, which assumes that forces are transferred only vertically between pair of consecutive rings of stone blocks. According to this hypothesis, the analysis of limit equilibrium configurations is performed by considering an infinitesimal meridian wedge of the dome, and by balancing the stabilizing moments and the overturning moments related to the overhanging masses. Thus, the transmission of forces between stones along parallels, due to interlocking and friction, is not taken into account. Consequently, this approach consider the structure as a set of independent corbelled meridian arches, without taking into account the three-dimensional behavior of the whole structure, coming from the interactions of the meridian arches in the parallel transverse direction.

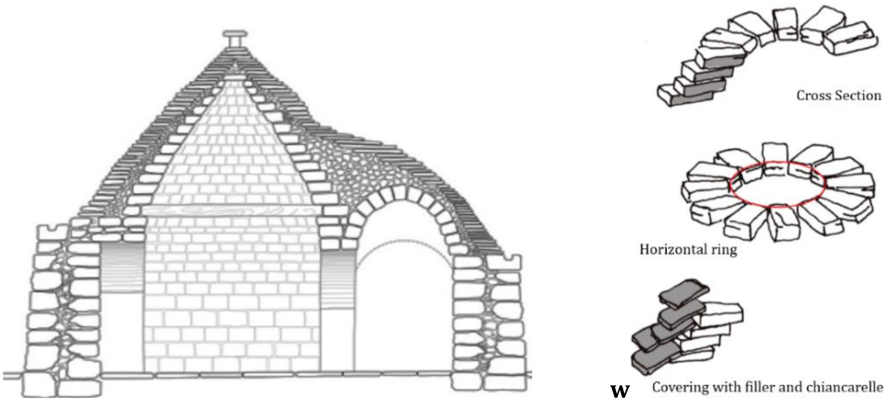


Fig. 1 Constructive section and detail of an Italian *Trullo* [14]

In [3], the Corbelling Theory is applied to the study of corbelled arches, tholoi and *Trulli*. The strong simplifications made by the Corbelling Theory involve significant shortcomings in view of an accurate description of the actual structural behavior of corbelled constructions. The limitations of the Corbelling Theory are emphasized in [8], where the cooperation among stones in the horizontal parallel rings is shown to be crucial for the interpretation of the mechanical behavior of corbelled domes.

Actually, for the *Trullo* case, and more generally for dry stone constructions, horizontal rings of stones are capable to bear compression along parallels, and thus to resist to inward overturning forces. This capacity is fully deployed also when the construction of the ring is not completed, thanks to the friction between lateral surfaces of adjacent courses. A practical evidence of this transmission of forces along parallels, and thus of the effectiveness of this constructive technique, is provided by several example of partially collapsed domes, showing that a meridian wedge is still able to stand without the collaboration of the collapsed surrounded structure (Fig. 2).

Recently, Rovero & Tonietti [19] proposed for domes with horizontal layers a Modified Corbelling Theory which is based on equilibrium equations that are capable of taking into account also the horizontal collaboration among adjacent infinitesimal wedges, due to the friction among blocks. This approach is then more accurate for describing the mechanics of corbelled vaults than the basic Corbelling Theory. The obtained results show that it is possible to further improve the theory in order to get more accurate predictions of the actual structural behavior of these structures.

Fig. 2 Example of partial collapse of *Trullo* vault [1]

3. Case Study

In view of the above considerations, we study the structural behavior of *Trulli* vaults by applying two different approaches to a case study represented by the corbelled dome of a rural *Trullo* (Fig. 3). In particular, the examined *Trullo* is located in Alberobello, a city of Valle d'Itria, in the middle of Apulia, UNESCO site since 1996.

Fig. 3 View of case study *Trullo* in Alberobello [20, 21]

The dome is built on a well-clamped squared basement made of irregular stones, by progressive superposition of jutting rings of dry stones without using mortar. The stones are fairly irregular, but they are cut slantwise along the intrados to obtain a continuous surface at the inner side of the dome. It covers a span of 3.84 m, while the thickness of the structural layer (*candela*) could be considered approximately constant and equal to 0.26 m.

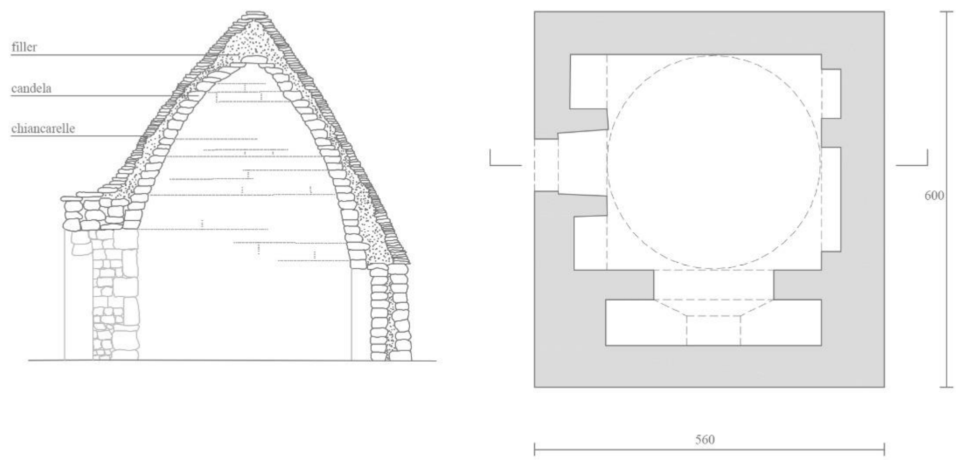


Fig. 4 Section [15] and plan of case study *Trullo* in Alberobello

4. Thrust Network Analysis of a *Trullo* Dome

4.1 Preliminary assumptions

Heyman [11] developed a Limit Analysis approach valid for masonry structures starting from the following constitutive assumptions on the masonry behavior (see also [9]):

(i) masonry is incapable of withstanding tensions;

(ii) masonry has infinite compressive strength;

(iii) sliding failure does not occur.

Under the abovementioned constitutive assumptions, it is possible to extend the Lower Bound Theorem of plasticity to masonry structures. In particular, in the context of masonry arches, the Lower Bound Theorem of plasticity can be formulated as follows: if it is possible to find a system of internal forces which is in equilibrium with the external loads and which corresponds to a thrust-line entirely contained in the thickness of the masonry, then the arch will not collapse under the applied loads. The theorem can be easily extended to the case of masonry vaults by substituting the thrust-line with a thrust-surface entirely contained in the thickness of the vault.

4.2 The classic thrust line analysis

Classical approaches for equilibrium analysis like the thrust line analysis are very effective for determining the structural safety of ordinary two-dimensional masonry constructions. However, aforementioned approaches do not apply for corbelled structures because of their particular construction technique, and then of their particular structural behaviour. For example, in the case of *Trulli* and by considering the actual dead load distribution for the dome (candela, filler and *chiancarelle*), it is not possible to find a thrust line contained within the thickness of the dome (Fig. 5). Therefore, the thrust line analysis would seem to lead to the wrong conclusion that this kind of structures should never be "safe".

Actually, the structural safety of corbelled structures can be justified only by taking into account the three-dimensional structural behavior, that is by considering the transmission of horizontal forces through annular load paths.

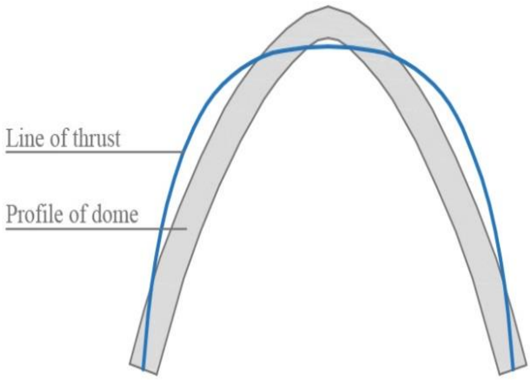


Fig. 5 Line of thrust and the structural profile of *Trullo*

4.3 Fully three-dimensional equilibrium analysis

In order to extend the concept of the thrust line analysis also to spatial structures, O'Dwyer [16] introduced the use of 3D funicular force networks first defined in plan. Using optimization tools, spatial compression-only networks that are in equilibrium with the self-weight and the applied loads, and that fit within the geometry of structure, could be obtained.

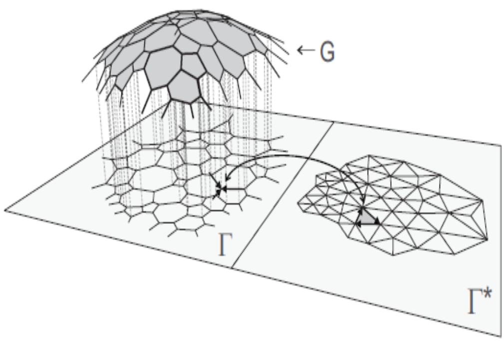


Fig. 6 Relationship between compression shell (G), its planar projection (primal grid Γ) and the reciprocal diagram (dual grid Γ^*) to determine equilibrium [5].

Afterwards, Block & Ochsendorf [5] extended the O'Dwyer's approach introducing the Thrust Network Analysis (TNA), that uses the reciprocal force diagrams of graphic statics for describing the possible horizontal equilibria, in order to obtain compression-only spatial networks in equilibrium with the self-weight. The integration between the TNA method and optimization algorithms allows for finding possible equilibrium networks only in compression within the thickness of masonry vaults.

The main steps of the TNA method by Block & Ochsendorf [5] are:

a) Defining the structure geometry: the solution G (compression-only network in equilibrium with the applied loads) must lie within given boundaries defined by the internal and external surfaces of the masonry construction.

b) Defining the primal grid Γ, i.e., a possible force pattern topology in the plan projection of the structure (Fig. 6). The primal grid Γ is the horizontal projection of the thrust network representing the solution G.

c) Assigning the loads: only vertical loads are attributed to the nodes proportionally to tributary area of each node with respect of the primal grid Γ.

d) Generating the force diagram, i.e., the dual grid Γ^*. Γ^* is generated from the primal grid Γ according to Maxwell's definition of reciprocal figures: corresponding branches are parallel, and the nodal equilibrium in the primal grid Γ is guaranteed by the presence of closed polygons in the dual grid Γ^*. The applied loads do not appear in the dual grid Γ^* because they degenerate into points in the horizontal projection. Therefore, the dual grid Γ^* has an unknown scale ζ since the relation between Γ and Γ^* holds regardless of their relative scales (Fig. 6).

e) Modifying the internal flow of forces: in the case of an indeterminate primal grid Γ, i.e., a grid with nodes connected with more of 3 other nodes, it is possible to change the internal force distribution by manipulating the force diagram.

f) Solving for determining the final result G: known the geometry of both the primal grid Γ and the dual grid Γ^*, the loads applied to the nodes and the boundary conditions, the problem can be solved using a one-step linear constrained optimization method.

4.4 Application of the TNA to a *Trullo* dome

For the analysis presented in this Section, we adopt the implementation of the TNA method in RhinoVault [17], a funicular form finding plug-in of the architectural CAD parametric software Rhinoceros©, aimed at the design of freeform thin shells. In order to extend the capability of RhinoVault to the search of equilibrium thrust networks in masonry-vaulted structures of given geometry, we used Python-scripts and an Evolutionary Optimization Algorithm implemented in Grasshopper, a visual programming language for Rhinoceros©. In particular, the Optimization Algorithm allows for fitting the thrust network within the thickness of the masonry vault. The adopted procedure is characterized by a clear visualization of the results and by an easy and fast interaction with the analysis process.

When the optimization algorithm produces a result, the ratio between the actual vault thickness and the thickness of the thinnest possible vault geometry enveloping the funicular solution can be considered as a Geometrical Factor of Safety (GFS) [12], which synthetically gives a measure of the structural safety of the vault.

The algorithm gives automatically a lower-bound of the maximum Geometric Factor of Safety (GFS) value for the corresponding equilibrium solution.

The main steps of the algorithm based on the Thrust Network Analysis (TNA) are summarized in [10].

The used primal grid Γ, represented in Fig. 6a, has 120 edges and 72 nodes. By the application of the TNA method, thanks to the manipulations of the dual force diagram Γ* and using an Evolutionary Optimization Algorithm, it was possible to find lower-bound solutions for each of the following load cases:

a) only the self-weight;

b) the dead loads (the self-weight and the weight of the infill);

c) the dead loads and a single eccentric point load of 5 kN.

We analyzed the effect of the single point load for each of nodes of the network, in order to incorporate in the study possible singularities in the loading condition.

Tab. 1 summarizes the results in term of maximum and average vertical deviations of the nodes of the thrust network from the middle surface, denoted as $|z - z^M|_{max}$ and $|z - z^M|_{av}$, respectively, and in term of GFS, for each the above listed load cases. In particular, for what concerns the effect of a single eccentric point load, Tab. 1 refers to the solution obtained for the position of the point load that involve the maximum deviation of the thrust network from the middle surface, as shown in Fig. 7d.

a) b) c) d)

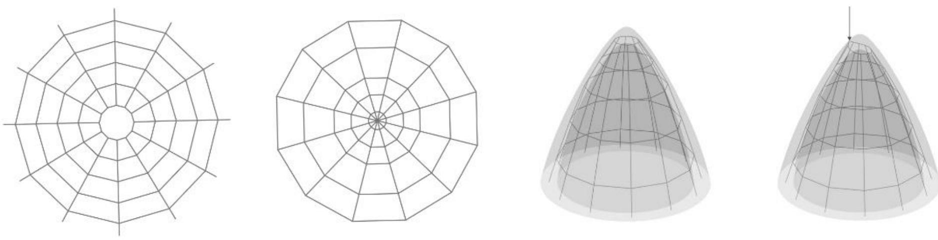


Fig. 7 Results of TNA analysis for dead loads: (a) Primal Grid *Γ*; (b) Dual force diagram *Γ*;* (c) 3D Thrust Network for dead loads; (d) 3D Thrust Network for dead loads and a single eccentric point load in the worst position for the structural safety of the vault

Under the self-weight, the thrust network fits the middle-surface of the vault remarkably well. The force diagrams clearly visualize the internal force distribution of the obtained solution (Fig. 7b). Notice that the addition of the load of the infill does not substantially vary the solution. The main vertical deviation occurs at the top of vaults, and it is limited to the *0.2 %* of the span of the dome.

Table 1 Results of the analysis in different loading condition

Load	Results		
	$\|z\text{-}z^M\|_{max}$ (mm)	$\|z\text{-}z^M\|_{av}$ (mm)	GFS
a	81	31	3.21
b	102	45	2.84
c	157	84	1.58

Finally, the load condition including a single eccentric point load is a strategy for assessing the safety of the vault under general live loads. We analyzed the effect of a 5 kN single point load in various nodes of the network, and we obtained always as solutions thrust networks contained into the thickness of the masonry.

Anyway, differently from the case of ordinary masonry vaults, TNA analysis may have a significant drawback for corbelled vaults. Indeed, since the joints between blocks are horizontal, the shear action of the horizontal component of meridian forces can produce outward horizontal sliding of the courses, which is opposed by the friction between the blocks [9].

The coefficient of static friction f is 1.5 to 3 times greater than the coefficient of sliding friction. The latter, for stone on stone, can be approximately assumed equal to 0.75; thus the coefficient of static friction f is at least equal to 1.125 [9]. By the relation f = tanϕ, we then find that the static friction angle ϕ is equal to 48,37°.

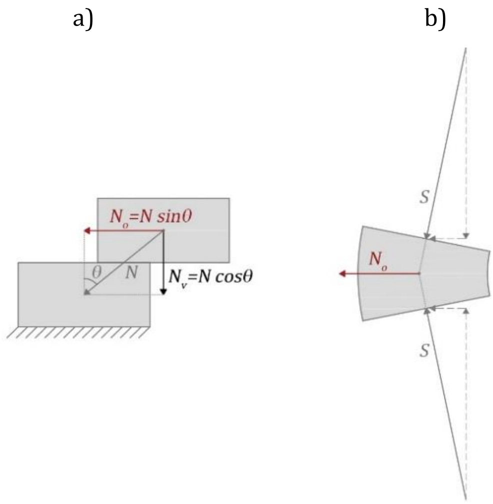


Fig. 8 Two-blocks scheme of the three-dimensional structural behavior of a corbelled dome

If we denote by θ the angle between the thrust network determined by the TNA analysis and the vertical plane (Fig. 8), sliding does not occur if

$$N\,sin\theta \leq f\,N\,cos\,\theta, \tag{1}$$

that is if

$$tan\,\theta \leq tan\,\phi. \tag{2}$$

Thus, condition (2) simply implies that in order to avoid the sliding between blocks, the angle θ between the thrust network branches and the vertical plane has to be smaller (at least, equal) than the friction angle ϕ.

Since, it may happen that, in general, the angle between the thrust network branches and the vertical plane is greater than the static friction angle, the solution determined by TNA analysis may be not compatible with one of the fundamental hypotheses of the Lower

Bound Theorem for masonry structures. Indeed, for the validity of the theorem it is required that sliding between blocks does not occur.

Therefore, the validity of the solution obtained by TNA analysis should be a posteriori verified: if the angle between the thrust network branches and the horizontal planes is greater than the friction angle, the solution is meaningless.

For the case study, the obtained TNA solution is admissible because the angles with respect to the vertical plane of all the meridian branches of the thrust network are lower than 48,37°.

5. Static Assessment of a *Trullo* Dome by New Formulation of the Modified Corbelling Theory

Although computational methods like TNA represent a fast design-oriented tool for the analysis of corbelled domes, the limitations underlined at the end of the previous Section lead us to search for other approaches, specifically dedicated to the structural analysis of corbelled domes, and capable of understanding the peculiar behavior of these structures.

A review of the (scarce) literature on this subject shows that the more advanced method available in the literature is, at our knowledge, that proposed by Rovero & Tonietti in [19], which in turn is based on the so-called Corbelling Theory by Benvenuto & Corradi [3].

5.1 The classic corbelling theory and the modified corbelling theory

The Classic Corbelling Theory, and in particular the model proposed by Benvenuto & Corradi [3], is based on the limit overturning equilibrium of a corbelled structure by considering as unknowns the functions describing the shape of the intrados and of the extrados of the vault.

Assuming that a false dome behaves exclusively as a polar series of false arches, each infinitesimal slice should be able to sustain itself and it should be statically independent from the others. By accounting for this and by considering the following assumptions:

(i) rigid blocks;

(ii) perfectly horizontal layers;

(iii) no mortar and infinitesimal distance between overlapping layers;

(iv) sliding between blocks does not occur;

the intrados and extrados profiles of the dome can be defined by continuous function and the equilibrium between the overturning moment (M_R) and the stabilizing moment (M_S) can be described using a differential formulation.

In this way, a complicated set of integral equations can be obtained, and closed-form solutions are possible only using several approximations, in particular in the expression of the stabilizing moment (M_S).

In [19], Rovero & Tonietti – after an in-depth review of the Classic Corbelling Theory in [3] – applied this theory to two paradigmatic case studies of corbelled domes. The comparison between theoretical and actual profiles clearly showed a significant deviation. The Authors of [19] ascribe the fail of the standard Corbelling Theory to the fact that this theory disregards the actual three-dimensional structural behavior of corbelled constructions that, as already underlined in [8], seems to be crucial for justifying the stability of this kind of structures.

In order to overcome the highlighted limitations of Classic Corbelling Theory, Rovero & Tonietti developed in [19] a Modified Corbelling Theory. This approach is based on that in [3], but is also capable to taking into account the collaboration between meridian wedges due to the force transmission along parallels, that are allowed by the interlocking and the friction between stone blocks.

To this aim, meridian wedges with finite width are considered (their dimensions depend on a parameter φ representing the width of the meridian wedge), reproducing the continuity along parallels, and thus taking into account the transversal static cooperation among adjacent infinitesimal meridian wedges. The other hypotheses of the theory in [19] are the same of that of the theory in [3]; in particular, both theories considers an analogous simplification on the geometry of the region of the wedge responsible of the stabilizing contribution in the overturning equilibrium equation.

With the aid of some numerical experiments, it is shown in [19] that the Modified Corbelling Theory is capable of much better results with respect to the Classic Corbelling Theory in [3], although the more complex mathematical formulation prevents the finding of closed-form solutions, and renders the problem essentially only numerically tractable.

In particular, using the approach in [19], the solution depends on the angular width φ of the meridian wedge. For the "optimal" value of φ, it is possible to obtain theoretical profiles that are closer to the actual profiles of the *Trullo* dome than in [3]. The Authors underline that the "optimal" value of φ conceptually expresses the minimal extension of the static cooperation along the horizontal rings necessary to achieve the limit equilibrium state, and that the obtained results in terms of φ are confirmed by the ruins of several partial collapses of corbelled domes as in Fig. 2.

5.2 A new formulation of the modified corbelling theory

Whereas the approach in [19] represents a great improvement over the approach in [3], the quality of the results could be still subject to further enhancements, as indicate the deviations still existing between the actual profile of the vault and the profile obtained by the Modified Corbelling Theory (see Fig. 19 and 20 of [19]).

In this vein, we observe that the approximation on the geometry of the region of the wedge responsible of the stabilizing contribution introduced in [3] produces an error which is negligible only in the assumption of infinitesimal wedge (as in [3]), but which increases proportionally with the angle φ in the assumption of finite dimension of the wedge (as in [19]).

Consequently, in order to further improve the theory, we have modified the approach in [19] by proposing a new formulation of the problem in which both the overturning moment (M_R) and the stabilizing moment (M_S) are expressed in an integral form. With reference to Fig. 9, and by considering a finite-dimension wedge of the dome of width φ, for any point *P(x)* belonging to the intrados curve *y(x)*, two region are defined: *PQR*, whose mass is related to the stabilizing moment *(M_S)* and *PROA*, whose mass generates the overturning moment (M_R).

As in [19], the assumption of finite-size meridian wedges imply that the overturning axis moves from the point *P* (position of the overturning axis in the assumption of infinitesimal wedge) toward the center of the dome to a point *P** (Fig. 9), thereby increasing the stabilizing moment (M_S) with respect the overturning moment (M_R).

For evaluating the expressions of the overturning moment M_R and of the stabilizing moment M_S, the width φ determines the distances from the rotation axis of the centroids of the infinitesimal slices subdividing the region *PROA* and of the infinitesimal slices subdividing the region *PQR*:

$$M_R = \gamma \int_0^x \big(y(\xi) - Y(\xi) \big) \left(x \cos\frac{\varphi}{2} - \xi \right) \big(\xi\varphi \big) d\xi, \tag{3}$$

$$M_s = \gamma \int_x^{x+h} \big(y(x) - Y(\xi) \big) \left(\xi - x \cos\frac{\varphi}{2} \right) \big(\xi\varphi \big) d\xi, \tag{4}$$

where *y(x)* and *Y(x)* are the ordinates of *P* and of a corresponding point *R* on the extrados; γ is the specific weight; and *h(x)* is the thickness of the dome at *x*.

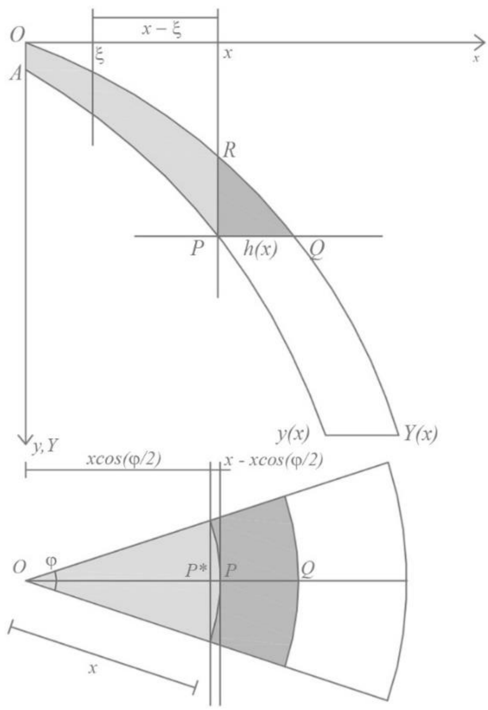


Fig. 9 Finite dimension meridian wedge in plan and section

The balance equation between the above mentioned moments can be written in the form:

$$M_S = \rho M_R, \tag{5}$$

where ρ is a suitable safety coefficient.

By assuming constant thickness $h = h_0$ and $\rho = 1$, and in view of (3)-(4), (5) takes the form:

$$\int_0^x \left(y(\xi)-Y(\xi)\right)\left(x\cos\frac{\varphi}{2}-\xi\right)\left(\xi\varphi\right)d\xi + \\ -\int_x^{x+h_0}\left(y(x)-Y(\xi)\right)\left(\xi-x\cos\frac{\varphi}{2}\right)\left(\xi\varphi\right)d\xi = 0. \tag{6}$$

By differentiating two time with respect to *x*, (6) can be reduced in the following differential form:

$$-\frac{1}{6}h_0\varphi\left\{\begin{array}{l} -6\left[-h_0-2x+(h_0+3x)\cos\frac{\varphi}{2}\right]y'(x)+ \\ \left[2(h_0^2+3h_0x+3x^2)-3x(h_0+2x)\cos\frac{\varphi}{2}\right]y''(x) \end{array}\right\}=0. \tag{7}$$

Furthermore, the thickness *h(x)* can be express as follow:

$$h(x)=\frac{y(x)-Y(x)}{Y'(x)}, \tag{8}$$

and the boundary conditions can be imposed for $x = 0$, i.e. at dome keystone, in the form:

$$Y(0)=0; \quad y(0)=m; \quad y'(0)=n, \tag{9}$$

where *m* and *n* are parameters depending by geometry of the dome.

Like for the approach in [19], also for our approach it is practically impossible to find closed-form solutions, and also the new procedure here proposed has to be numerically performed. In particular, the system of differential equations (7) and (8) with the boundary condition (9) can be solved numerically for obtaining as solutions the shape functions *y(x)* and *Y(x)* of the intrados and the extrados of the vault, respectively. Of course, the obtained solution depends on the parameter φ.

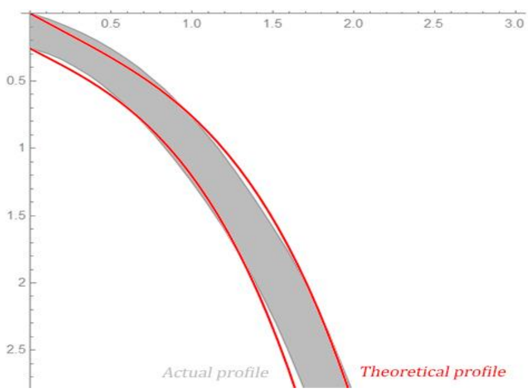


Fig. 10 Actual and theoretical profiles obtained by the new formulation of the Modified Corbelling Theory

We implemented the described analytical model in the software Wolfram Mathematica©, in order to find numerical solutions of the differential equations governing the limit equilibrium of the corbelled vault of the *Trullo* under investigation (see Sect. 3). In this case, for the boundary conditions we assume $m = 0.26$ and $n = m/h$. Since the solutions depend on the angle φ, we search for the value of φ which minimizes the maximum horizontal distance between the actual axis of the dome and the theoretical profile, i.e., $\varphi = 60.8°$. For this angle, by applying the described approach we obtain the two limit curves of the intrados $y(x)$ and the extrados $Y(x)$ which are compared in Fig. 10 to the actual profiles of the vault. As it is clearly shown by Fig. 10, by our approach we obtain theoretical profiles of the vaults practically coincident with the actual profiles.

For the sake of a comparison, we evaluated for the same *Trullo* described in Sect. 3 and for the same boundary conditions the theoretical profiles obtainable by our approach (the new formulation of the Modified Corbelling Theory), by the approach proposed in [19] (the Modified Corbelling Theory), and by the approach proposed in [3] (the Classic Corbelling Theory).

As it is shown in Fig. 11a, our approach allows for a better approximation of the actual shape of the vault with respect to the approach proposed in [19], which in turn improves the approach proposed in [3]. Notice that the profiles compared in Fig. 11a are obtained in correspondence of the values of the angular width of the meridian wedge φ which allow for the best fitting between theoretical and actual profile of the corbelled vault. In particular, the best fitting curves represented in Fig. 11a are obtained for $\varphi = 60{,}8°$ for our approach, and for $\varphi = 69°$ for the approach in [19] (the theoretical profiles are independent on φ for the Classic Corbelling Theory in [3]).

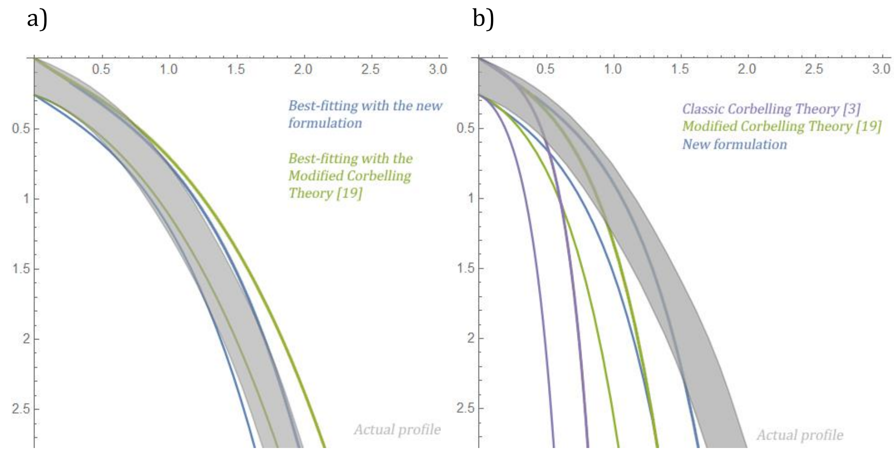


Fig. 11 Comparison between the Classic Corbelling Theory, the Modified Corbelling Theory and the new formulation of the Modified Corbelling Theory: (a) best-fitting solutions; (b) solutions for $\varphi = 50°$ (notice: for the Classing Corbelling Theory the solution do not depend on φ)

Besides the capability of better approximating the actual profile of the corbelled vault, our approach improves that in [19] also because the best fitting between actual and theoretical profiles of the vault is obtained for considerably smaller values of the angle φ than those necessary for the approach in [19]. We recall that, as stated in [19], the values of φ are related to the structural efficiency of the dome, and smaller values of φ are related to more efficient shapes of the dome. Thus, in the spirit of Limit Analysis our procedure, giving the smaller value of φ, gives also a better estimate of the actual load-bearing capacity of the structure. In other words, our procedure produces less conservative results than that in [19].

In order to clarify the above sentence, we show in Fig. 11b a comparison between the theoretical profiles obtainable by the three above mentioned approaches for the same value of φ , and in particular for $\varphi=50°$ (again, we point out that the theoretical profiles are independent on φ for the Classic Corbelling Theory in [3]). Fig. 11b enlighten that for $\varphi=50°$ the profiles obtained by our approach are much more closer to the actual shape of the vault than those obtained by the other considered approaches.

Finally, we observe that the computational efforts required by our approach are very similar than those required by the approach in [19]: indeed, the time elapsed by the software Wolfram Mathematica© for performing the entire calculation with our approach and that in [19] is *0.03s* and *0.02s*, respectively. Thus, the increase in the computational costs coming from the removal of the approximation contained in [19] is negligible.

6. Conclusions

In this paper we have studied new computational and analytical approaches for assessing of the structural safety of corbelled vaults. The computational approach consists in the application of the Thrust Network Analysis (TNA), together with suitable optimization procedures. The analytical approach is an improvement of the Modified Corbelling Theory in [19]. In both cases, the crucial aspect is that of representing in the analysis the actual three-dimensional structural behavior of these peculiar masonry constructions. The considered case study, an Alberobello *Trullo*, represent a paradigmatic and relevant case of corbelled vault construction belonging to the Italian cultural heritage.

The TNA – yet employed for the limit analysis of ordinary masonry vaults, and here extended also to the case of corbelled vaults – reveals a fast and effective approach. The three-dimensional structural behavior, involving the stabilizing effect of the collaboration between adjacent meridian wedges, is represented by TNA in term of forces acting along the horizontal rings. For the case study under investigation, TNA was successfully and easily employed for a sensitivity analysis about the effect of an eccentric load with variable position on the structural safety of the vault. Moreover, this computational method could be easily extended to load conditions more complex than those here considered. Anyway, we observe that this approach may have a significant drawback for corbelled vaults. Indeed, since the joints between blocks are horizontal, the obtained solution is meaningful only if the angle between the thrust branches determined by TNA analysis and the horizontal planes is smaller than the friction angle. Thus, the application of TNA to corbelled vaults require to be a-posteriori verified.

On the other hand, analytical approaches like the Corbelling Theory, expressly developed for corbelled structures, are capable of a better description of the mechanics of these particular constructions, although they requires a greater burden from the mathematical point of view. Here, starting from the Modified Corbelling Theory in [19], we propose a further improvement of the theory, capable of definitely better results.

In particular, our new formulation of the Modified Corbelling Theory allows for obtaining theoretical profiles closer to the actual profile of the vault than those obtainable by the procedure in [19], without any substantial increase of the complexity in the model or of the computational costs. Moreover, our procedure gives the best fit solutions for smaller angles φ (identifying the finite width of the meridian wedge capable to verify the overturning equilibrium) than the procedure in [19]. Thus, in the spirit of Limit Analysis, our approach allows for less conservative results than that in [19].

References

[1] Allen E. Pietre di Puglia: Dolmen, Trulli e Insediamenti Rupestri, Adda Editore, Bari, Italy, 1984.

[2] Benvenuto E. La Scienza delle Costruzioni e il suo Sviluppo Storico, Sansoni Editori, Firenze, Italy, 1981.

[3] Benvenuto E, Corradi M. La statica delle false volte. Architettura in pietra a secco. Schena Editori, 1987, 93-106.

[4] Block P, Ochsendorf J. Thrust network analysis: a new methodology for three-dimensional equilibrium. Journal of the IASS, 2006; 48: 167-173.

[5] Block P. Thrust network analysis: exploring three-dimensional equilibrium. Ph.D. Dissertation, Massachusetts Institute of Technology, Cambridge, 2009.

[6] Block P, Lachauer L. Three-dimensional (3D) equilibrium analysis of gothic masonry vaults. International Journal of Architectural Heritage, 2014; 8: 312-335. https://doi.org/10.1080/15583058.2013.826301

[7] Cavanagh WG, Laxon R.R. The structural mechanics of the Mycenaean Tholos Tomb. Annu Brit Sch Athens, 1981; 109-140.

[8] Como MT. Analysis of the statics of Mycenaean Tholoi. Proceedings of the 2th International Congress on Construction History, Cambridge, UK, 2006; 777-790.

[9] Como M. Statics of Historic Masonry Constructions, Springer-Verlag Press, Berlin, Germany, 2013.
https://doi.org/10.1007/978-3-642-30132-2

[10] Foti P, Fraddosio A, Lepore N, Piccioni MD. Three-dimensional lower-bound analysis of masonry structures. Structures and Architecture: Beyond their Limits. CRC Press, 2016, 558-566.
https://doi.org/10.1201/b20891-77

[11] Heyman J. The stone skeleton. International Journal of Solid and Structures, 1966; 2: 249-279.
https://doi.org/10.1016/0020-7683(66)90018-7

[12] Heyman J. The Stone Skeleton: Structural Engineering of Masonry Structures, Cambridge University Press, Cambridge, UK, 1995.
https://doi.org/10.1017/CBO9781107050310

[13] Huerta S. Mechanics of masonry vaults: the equilibrium approach, Proceedings of Historical Constructions, Guimaraes, Portugal, 2001; 167-173.

[14] Mecca S, Dipasquale L. Earthen domes et habitats: villages of northern Syria, Edizioni ETS, Pisa, Italy, 2009.

[15] Moramarco V. I Trulli di Alberobello: Lettura Costruttiva e Indagine Statica, Fasidiluna Editore, Bari, Italy, 2013.

[16] O'Dwyer DW. Funicular analysis of masonry vaults. Computer & Structures, 1999; 73: 187-197.
https://doi.org/10.1016/S0045-7949(98)00279-X

[17] Rippmann M, Lachauer L, and Block P. Interactive vault design. International Journal of Space Structures, 2012; 27: 219-230.
https://doi.org/10.1260/0266-3511.27.4.219

[18] Rovero L, Tonietti U. Structural Behavior of Earthen Corbelled Domes in the Aleppo's region. Materials and Structures, 2012; 45:171–184.

https://doi.org/10.1617/s11527-011-9758-1
[19] Rovero L, Tonietti U. A modified corbelling theory for domes with horizontal layers. Construction and Building Materials, 2014; 50: 50-61. https://doi.org/10.1016/j.conbuildmat.2013.08.032
[20] Todisco L, Sanitate G. Static stability of trulli. Materials and Structures, 2015; 49: 2893-2905. https://doi.org/10.1617/s11527-015-0693-4
[21] Todisco L, Sanitate G, Monti G. Database of the Apuliabase (2014) Project. http://www.databaseapuliabase.altervista.org//.