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# Research on Engineering Structures & Materials





Research Article

# Functionally graded Timoshenko beams with elastically-restrained edge supports: thermal buckling analysis via stokes' transformation technique

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Article Info	Abstract					
Article history Received 18 Oct 2016 Revised 1 April 2017 Accepted 5 May 2017 Keywords: Thermal buckling, Timoshenko beam theory, Functionally graded material, Stokes' transformation	The present study investigates buckling in functionally graded material (FGM) beams when exposed to a temperature rise. The proposed FGM beams have arbitrary edge supports that are modeled by rotational and translational springs. The mechanical properties are assumed to vary continuously across the thickness direction according to a simple four-parameter power law. To obtain the critical value of temperature, the governing equilibrium equations are extracted based on Timoshenko beam theory, using the assumption of Von-Karman nonlinearity for the physical neutral surface concept. The equations are further solved by Fourier series expansion via Stokes' transformation technique. Numerical examples are provided to demonstrate the accuracy and reliability of the proposed method. The influence of two models of metal-ceramic distribution across the thickness (symmetrical and unsymmetrical ones) on the response of the beam in thermal buckling of FG beam is investigated. It is observed that, the critical buckling temperature rises more for symmetrical model of FGM beam with respected to unsymmetrical one. Also, increasing the translational and rotational spring coefficient makes the beam stiffer; consequently, the critical buckling temperature is the proposed for the proposed for the proposed for the proposed for the beam stiffer; consequently, the critical buckling temperature is the beam stiffer; consequently, the critical buckling temperature is the proposed for the proposed for the proposed for the beam stiffer; consequently, the critical buckling temperature is the proposed for the proposed for the proposed for the proposed for the beam stiffer; consequently, the critical buckling temperature is the proposed for the beam stiffer; consequently, the critical buckling temperature is the proposed for the proposed for the proposed temperature is the proposed for the pro					
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# 1. Introduction

In recent years, several research activities have been motivated by considerable attention to functionally graded materials (FGMs). FGMs are a branch of new composite materials which are employed to design structures when exposed to high temperatures. A review of the available literature reveals the importance of investigations on the thermal buckling phenomenon of functionally graded beams. For example, Li et al. [1] presented the analysis of thermal post-buckling of FGM Timoshenko beams subjected to transversely non-uniform temperature rise by applying the shooting method. Song and Li [2] inspected the thermal buckling and post-buckling of pinned-fixed Euler-Bernoulli beams resting on an elastic foundation based on the accurate geometrical nonlinear theory by considering the effects of both linear and nonlinear elastic foundations. Aydogdu [3] studied the thermal buckling of cross-ply laminated composite beams under different sets of boundary conditions based on TSDT (Third order of Shear Deformation Theory) via Ritz method. Shahsiah et al. [4] proposed the thermal buckling of FG beams based on the one-dimensional theory of elasticity by employing Tanigawa's model for the variation of Poisson's ratio, the modulus of shear stress, and the coefficient of thermal expansion. Furthermore, Kiani and Eslami [5] analyzed the thermal buckling of FG beams based on Euler-Bernoulli theory in three types of thermal loading across the thickness for different boundary conditions. Miraliari [6] examined the

effects of various edge conditions on the thermal buckling of FG beams based on TSDT with and without piezoelectric layers. Wattanasakulpong et al. [7] investigated the thermal buckling and elastic vibration of FG beams based on HSDT (Higher order of Shear Deformation Theory). They examined Ritz method to study the influence of boundary conditions on critical buckling temperature. Zhang [8] proposed the numerical solution by applying Ritz method to the nonlinear bending of FGM beams. He derived the governing equations based on the physical neutral surface concept, HSDT, and nonlinear Von-Kármán strain-displacement relationships. In the other work, based on the previous article, Zhang [9] proposed the thermal post-buckling and nonlinear vibration behaviors of FGM beams by using the concept of physical neutral surface. He employed Ritz method to solve the governing equations to investigate the influence of various boundary conditions on critical temperature and frequency. Kiani and Eslami [10] proposed the thermomechanical buckling behavior of temperature-dependent FGM beams based on FSDT (First order of Shear Deformation Theory) for Timoshenko beams under different boundary conditions for different temperature distributions. Sun et al. [11] discussed the buckling and post-buckling thermomechanical deformations of FGM Timoshenko beams resting on a two-parameter, nonlinear, elastic foundation by applying the shooting method.

Calim [12] carried out the transient analysis of axially functionally-graded (AFG) Timoshenko beams with variable cross-sections. By deploying the complementary functions method (CFM) and the modified Durbin's algorithm, he concluded that the material inhomogeneity, taper parameter, and the assumed boundary conditions have significant effects on the dynamic response of AFG-tapered Timoshenko beams. Simsek [13] investigated the buckling of Timoshenko beams composed of the two-dimensional functionally-graded material with different boundary conditions. In an excellent monograph by Wang et al. [14], a wide range of solutions is presented for the buckling problems of beams, plates, and shells. Avdogdu [15] used the Ritz method to examine the buckling behavior of the laminated composite beams based on various beam theories. In another research, Aydogdu [16] studied the thermal buckling analysis of the cross-ply laminated beams under different sets of boundary conditions by using a three-degree freedom shear deformable beam theory on the basis of Ritz method. Deng et al. [17] presented the exact solutions of double-functionally-graded Timoshenko beam system on Winkler-Pasternak elastic foundation, which are benchmarks of double-beam systems in the field of engineering. In addition, they derived the motion differential equations of the double-beam system by employing Hamilton principle. Nami et al. [18] investigated the thermal buckling analysis of functionally-graded rectangular nanoplates. By assuming the nonlocal elasticity theory and using the third-order shear deformation theory, they indicated that the critical temperatures rise by increasing the power law indices. Fallah and Aghdam [19] proposed the thermal buckling analysis for the uniform temperature rise of FGM beams on the elastic foundation by employing Euler-Bernoulli beam theory and considering temperature-independent (TID) material properties. Esfahani et al. [20] studied the thermal buckling and post-buckling behavior of FGM Timoshenko beams resting on the nonlinear hardening elastic foundation. Ghiasian et al. [21] investigated the static and dynamic buckling behaviors of FGM Euler-Bernoulli beams subjected to the uniform temperature rise and resting on the nonlinear elastic foundation. The nonlinear thermal post-buckling behavior of shear deformable stainless steel-silicon nitride beam resting on elastic foundation was studied by Shen and Wang [22] under uniform and nonuniform temperature rise. Ghiasian et al. [23] investigated the nonlinear thermal dynamic buckling behavior of FGM Timoshenko beams subjected to the sudden uniform temperature rise. Amlan and Debabrata [24] studied the nonlinear post-buckling load-deflection behavior of functionally-graded material (FGM) Timoshenko beam under in-plane thermal loading. They applied the thermal loading by providing non-uniform temperature rise across the beam thickness at the steady-state conditions. By employing Euler–Bernoulli beam theory, Fu et al. [25] studied the thermo-piezoelectric buckling and dynamic stability for the piezoelectric functionally-graded beams subjected to one-dimensional steady heat conduction in the thickness direction. Zhao et al. [26] examined the thermal post-buckling behavior of simply-supported thin FGM beams with temperature independent (TID) material properties under uniform and some special cases of non-uniform temperature rise. Heydari [27-28] presented a new analytical method for the buckling analysis of circular plates with constant thickness and Poisson's ratio, which were made of functionally-graded materials subjected to radial loading. He investigated the increase in the buckling capacity and the improvement in the behavior of functionally-graded plates in comparison with the homogeneous plates. Additionally, Heydari et al. [29] presented a numerical method for the buckling analysis of functionally-graded circular plates (FGCP) subjected to the uniform radial compression including shear deformation resting on Pasternak elastic foundation. They also studied the effects of both linear and quadratic thickness variations and Poisson's ratio on FGCP.

Although Fourier series expansion is utilized for the solution of a differential equation or partial differential equation, the term-by-term differentiation is not permissible; Stokes' transformation is a legal method confronting it. Stokes' transformation technique was first proposed by Stokes [30]. Although a few works have been published concerning this technique, the literature review reveals more acceptable correctness and convergence of the solution method compared to those of the other common approaches. In what follows, a number of studies is discussed on the application of Stokes' transformation technique. Chung [31] exploited Fourier series expansion in governing differential equation and via Stokes' transformation technique in order to evaluate the free vibration characteristics of a circular, cylindrical shell. Chuang and Wang [32] analyzed the effect of the vibration of axially loaded damped beams on the viscoelastic foundation by using the trigonometric series solution in conjunction with Stokes' transformation method. Yokoyama [33] analyzed the vibration of Timoshenko beam columns on the two-parameter elastic foundations. Chen et al. [34] considered the transient and random responses of structures subject to the motion support by Stokes' transformation technique. Kim and Kim [35] calculated the vibration characteristics of PWR fuel assembly with reactor end boundary conditions via Stokes' transformation method. In another research, Kim and Kim [36] calculated the vibration frequencies of beams with classical and non-classical boundary conditions by using Stokes' transformation method. Khalili et al. [37] evaluated the responses of laminated composite plates subjected to static and dynamic loadings with different boundary conditions by employing Stokes' transformation method. Shao and Ma [38] carried out the analysis of the free vibration of laminated cylindrical shells with arbitrary classical boundary conditions by means of Fourier series expansion. Ansari and Darvizeh [39] predicted the dynamic behavior of FGM shells under arbitrary boundary conditions through Stokes' transformation method. Latifi et al. [40] studied the buckling analysis of rectangular FG plates under various edge conditions by applying Fourier series expansion and Stokes' transformation technique.

This study is focused on determination of critical temperature at the onset of buckling in Timoshenko FG beams, which have two elastic constraints at their two ends while subjected to temperature gradient. By applying Stokes' transformation technique to the governing equations, the critical buckling temperature of the beam is obtained. Comparison of the numerical results with those reported in the previous studies is also presented. To the best knowledge of the authors, this work is the first attempt to investigate the thermal buckling behavior of FG beams via Fourier series expansion and Stokes' transformation technique for different boundary conditions. To this end, the rest of the paper is organized as follows: In section 2, the material properties of FGs are introduced based on two simple, four-parameter and power-law, distributions. In section 3, the governing equilibrium in thermal buckling is obtained through assuming Von-Karman nonlinear strain-displacement relations. In section 4, by expressing the trigonometric form of Fourier series expansion as the displacement field of the FG beam, Stokes' transformation technique is applied to the governing equations as well

as the imposed boundary conditions so as to determine the critical value of temperature. In section 5, some numerical results are provided to investigate the efficiency and accuracy of the proposed method in dealing with the thermal buckling phenomenon.

### 2. FG Material Constitutive Relations

Herein, a functionally-graded beam with the length of L and the thickness of h is considered, as shown in Fig. 1. The beam cross-section is rectangular; moreover, the beam is put on the xz plane, where the x-axis is parallel to the length of the beam, and the y-direction represents the width. The origin of the coordinate system is at the mid-surface. In addition, the top and bottom surfaces are at z=-h/2 and z=h/2, respectively.



Fig. 1. The FG beam geometry

In the current research, it is assumed that the FGM is made up of a mixture of ceramic and metal constituents. The mechanical and thermal properties of the FGM beam can be expressed by the following linear combination [41]:

$$P(z) = (P_c - P_m)V_c + P_m$$
(2.1)

where *P* and *V* denote the properties of FG material and volume fraction, respectively. The subscripts of *m* and *c* stand for the metallic and ceramic constituents, respectively. Poisson's ratio is considered to be constant across the thickness. It is worth noting that the volume fractions of the two constituent materials should add up to unity. The ceramic volume fraction  $V_c$  follows two simple, four-parameter, power-law distributions presented in [42]:

$$FGM_{1(a/b/c/p)}: V_{c} = \left[1 - a\left(\frac{1}{2} + \frac{z}{h}\right) + b\left(\frac{1}{2} + \frac{z}{h}\right)^{c}\right]^{p}$$

$$FGM_{2(a/b/c/p)}: V_{c} = \left[1 - a\left(\frac{1}{2} - \frac{z}{h}\right) + b\left(\frac{1}{2} - \frac{z}{h}\right)^{c}\right]^{p}$$
(2.2)

The volume fraction index p ( $0 \le p \le \infty$ ) and the parameters of a, b, and c dictate the material variation profile through the FG beam thickness. It is assumed that the material properties do not depend on temperature variation.

#### 3. Governing Equations

In order to study the thermal buckling behavior of FG beam in any combination of the boundary conditions, the beam is assumed to be elastically restrained by setting the rotational  $(K_2 \text{ and } K_4)$  and translational  $(K_1 \text{ and } K_3)$  springs at the two ends, as shown in Fig. 2.

According to the physical neutral surface, there is no stretching-bending coupling effect for the stress and stress-coupling resultants; therefore, the governing equations have the simple forms similar to those of the classical thin structure theory applied to homogeneous isotropic materials. We employed this approach in the present work. Based on the physical neutral surface concept, the distance between the middle surface and the physical neutral surface  $(z=z_0)$  is introduced in [9].



Fig. 2. Timoshenko beam with the elastically restrained ends

$$z_{0} = \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} zE(z)dz}{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)dz}$$
(3.1)

The concept of neutral surface is defined as the surface in which the normal stresses and consequently in-plane or axial strain equal zero. This concept can be followed in relatively thick plate too, regardless of what kind of the shear deformation theory is used. In the homogenous structure, this kind of surface coincide with middle surface, whereas in nonhomogeneous one such as FG and laminar composite plate that does not coincide with middle surface; therefore, it can be determined by formula (3-1) as mentioned in manuscript. The interested reader can find more details about this approach in [43] and [44]. By using Equation (3.1) and based on the assumption of Timoshenko beam, the displacement equations take the following form [45]:

$$U(x,z) = u(x) + (z - z_0)\varphi(x)$$

$$W(x,z) = w(x)$$
(3.2)

where *u* and *w* represent the displacement in the physical neutral surface along the coordinates of x and z, respectively. In addition,  $\varphi$  denotes the rotation around the y-axis (normal to xz plane) at z=z<sub>0</sub> of the deformed line. The strain components are as follows:

$$\{\varepsilon_x, \gamma_{xz}\} = \{\varepsilon_x^0, \gamma_{xz}^0\} + (z - z_0)\{\varepsilon_x^1, \gamma_{xz}^1\}$$
(3.3)

where  $\varepsilon_x$ ,  $\gamma_{xz}$  are normal and shear components of strain, respectively. The superscripts of 0 and 1 signify the strain and curvature in the physical neutral surface, respectively. By considering the nonlinear Von-Karman strain–displacement relationships, we can express the strains as follows [8]:

$$\{\varepsilon_x^0, \gamma_{xz}^0\} = \left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2, \varphi + \frac{dw}{dx}\right)$$
(3.4)

$$\{\varepsilon_x^1, \gamma_{xz}^1\} = \left(\frac{d\varphi}{dx}, 0\right) \tag{3.5}$$

As the modulus of elasticity varies across the thickness of the beam, we can express the stressstrain relations as follows by considering the linear thermo-elasticity [46]:

$$\begin{cases} \sigma_{\chi} \\ \tau_{\chi z} \end{cases} = \begin{bmatrix} \bar{Q}_{ij} \end{bmatrix} \begin{cases} \varepsilon_{\chi} - \varepsilon_{\chi}^{i} \\ \gamma_{\chi z} \end{cases} ; \quad i, j = 1,5$$
 (3.6)

where  $[\overline{Q}_{ii}]$  is the reduced stiffness matrix and is defined as follows [46]:

т.

$$\left[\bar{Q}_{ij}\right] = \frac{E(z)}{1 - v^2} \begin{bmatrix} 1 & 0\\ 0 & \frac{1 - v}{2} \end{bmatrix}$$
(3.7)

As seen in Eq. (3.7), we used the constitutive relations for FGM beams according to [46]; however, it is analytically showed in [47] that employing the reduced stiffness matrix which commonly is used for plate, yields identical results as those obtained when the beam analysis is carried out.

The forces and the moment per-unit-length are expressed in terms of the stresses by integrating Eq. (3.6) through the beam thickness; according to Timoshenko beam theory [48], they are expressed as follows:

$$N_{x} = A_{x} \left\{ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^{2} \right\} - N_{x}^{T}$$

$$M_{x} = D_{x} \frac{d\varphi}{dx} - M_{x}^{T}$$

$$Q_{xz} = \frac{A_{x}K_{s}}{2(1+v)} \left( \varphi + \frac{dw}{dx} \right)$$

$$(N_{x}^{T}, M_{x}^{T}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\alpha(z)\Delta T\{1, (z-z_{0})\}dz$$
(3.9)

where  $K_s$  denotes the shear correction factor, and  $N_x^T$ ,  $M_x^T$  are the thermal force and the moment, respectively. In addition,  $N_x$ ,  $Q_{xz}$  and  $M_x$  are the axial, shear forces, and moment-perunit length of the beam, respectively.  $A_x$ ,  $D_x$  and  $G^*$  are determined as follows:

$$(A_x, D_x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z)}{1 - v^2} (1, (z - z_0)^2) dz \quad ; \quad G^* = \frac{A_x K_s}{2(1 + v)}$$
(3.10)

To establish the equilibrium equations, the static version of the principle of virtual displacements is applied. According to this approach and in the absence of mechanical loading, a state of equilibrium occurs when the first variation of strain energy function vanishes. Hence, we can write:

$$\delta U = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx} \delta \varepsilon_{xx} + K_s \tau_{xz} \delta \gamma_{xz}) dx dz = 0$$
(3.11)

By substituting Eq. (3.4) and Eq. (3.6) into Eq. (3.11), and then, integrating it through the thickness with regard to Eq. (3.8), and setting the coefficients of  $\delta u$ ,  $\delta w$ , and  $\delta \varphi$  to zero separately, the equilibrium equations are obtained as follows:

$$\delta u: \frac{dN_x}{dx} = 0$$

$$\delta w: \frac{dM_x}{dx} - Q_{xz} = 0$$

$$\delta \varphi: \frac{dQ_{xz}}{dx} + N_x \frac{d^2 w}{dx^2} = 0$$
(3.12)

By substituting Eq. (3.8) into Eq. (3.12), the governing equilibrium equations in the buckling are derived as follows:

$$\frac{d^4w}{dx^4} - \lambda^2 \frac{d^2w}{dx^2} = 0 \quad ; \quad \lambda^2 = \frac{N_x G^*}{D_x (G^* + N_x)}$$
(3.13)

$$\varphi + \beta \frac{d^3 w}{dx^3} + \frac{dw}{dx} = 0, \qquad \beta = \frac{D_x}{G^*} \left( 1 + \frac{N_x}{G^*} \right)$$
(3.14)

#### 4. Application of Stokes' Transformation Technique

The beam lateral displacement w(x) and rotation  $\varphi(x)$  are assumed to have the trigonometric form of Fourier series. The lateral displacement function is defined in two separate regions: the boundary points and the intermediate regions between the boundary points. Moreover, the rotation function is defined for all points and expressed as follows:

$$w(x) = \begin{cases} \sum_{m=1}^{\infty} w_m \sin\left(\frac{m\pi x}{L}\right) &, \quad 0 < x < L \\ w_0 &, \quad x = 0 \\ w_L &, \quad x = L \end{cases}$$
(4.1)

$$\varphi(x) = \bar{\varphi}_0 + \sum_{m=1}^{\infty} \varphi_m \cos\left(\frac{m\pi x}{L}\right) \quad , \quad 0 \le x \le L$$

$$\varphi(0) = \bar{\varphi}_0 + \sum_{m=1}^{\infty} \varphi_m \tag{4.2}$$

$$\varphi(L) = \bar{\varphi}_0 + \sum_{m=1}^{\infty} \varphi_m (-1)^m$$

where  $w_m$  and  $\varphi_m$  are unknown coefficients of Fourier series for w and  $\varphi$  functions, respectively. The unknown coefficient  $\bar{\varphi}_0$  is assumed to be zero for some boundary conditions

such as the simply-supported edge. It is obviously clear from Eq. (4.1) that Fourier sine series has an appropriate form to satisfy the geometric and natural boundary conditions of a beam when both ends are simply supported. However, it is not necessary that the series satisfy all other boundary conditions since Stokes' transformation technique can be exploited for arbitrary boundary conditions with non-zero lateral displacement and edge moments. While the direct differentiation of a Fourier sine series leads to a cosine series without the constant term, it is not accomplished to be a complete set of functions. To obtain the correct series expressions for the derivatives of a Fourier series, Stokes' transformation technique must be employed. This technique defines each derivative with an independent series and obtains the newly defined series by integrating the parts in order to achieve the relationship between the Fourier coefficients (See Appendix A). This is the advantage of Stokes' transformation technique that we can define a Sinus Fourier Series (or Cosine Fourier Series) which is employed for all boundary conditions applicable for some degree of freedom in any arbitrary edges. for In order to employ Stokes' formulation in any combination of the boundary conditions, the structure is assumed to be elastically restrained by means of rotational and translational springs at the two ends of the beams and shells, four edges of the plates. In solving any boundary value problem, derivatives of the response function are needed to apply the geometric and natural boundary conditions. When a Fourier series is employed, it generally satisfies only one or a few particular sets of boundary conditions. Accordingly, for the problems with various unspecified boundary conditions, a particular derivation method different from the conventional techniques should be applied to be able to simultaneously deal with different boundary conditions; by employing Stokes' transformation, first- and higher-order derivate of a Fourier series are defined in a particular manner so that the derivative functions are related to the boundary values needed to obtain the response [41].

By substituting the derivatives of w(x) into Eq. (3.13), the unknown coefficient  $w_m$  can be determined, using the following equation:

$$\sum_{m=1}^{\infty} \gamma_m \left[ \frac{2}{L} \left( (w_L(-1)^m - w_0) (\lambda^2 + \gamma_m^2) + (w_0^{\prime\prime} - w_L^{\prime\prime} (-1)^m) \right) + \gamma_m w_m (\lambda^2 + \gamma_m^2) \right] \sin(\gamma_m x) = 0$$
(4.3)

Substituting the derivatives of  $\varphi(x)$  into Eq. (3.14) results in the following equation in terms of coefficients  $\varphi_m$  and  $\overline{\varphi}_0$ :

$$\begin{split} \bar{\varphi}_{0} + \sum_{m=1}^{\infty} \varphi_{m} \cos(\gamma_{m} x) + \beta \left\{ \frac{w_{L}^{\prime\prime} - w_{0}^{\prime\prime}}{L} + \sum_{m=1}^{\infty} \left[ \frac{2}{L} (w_{L}^{\prime\prime} (-1)^{m} - w_{0}^{\prime\prime}) - \gamma_{m}^{2} \left( \frac{2}{L} (w_{L} (-1)^{m} - w_{0}) + \gamma_{m} \cdot w_{m} \right) \right] \right\} \cos(\gamma_{m} x) + \frac{w_{L} - w_{0}}{L} + \\ \sum_{m=1}^{\infty} \left[ \frac{2}{L} (w_{L} (-1)^{m} - w_{0}) + \gamma_{m} \cdot w_{m} \right] \cos(\gamma_{m} x) = 0 \end{split}$$

$$(4.4)$$

Finally, the unknown coefficients  $w_m$ ,  $\varphi_m$  and  $\overline{\varphi}_0$  are demonstrated as follows:

$$w_m = \frac{2}{L} \frac{(w_L''(-1)^m - w_0'') + (w_0 - w_L(-1)^m)(\lambda^2 + \gamma_m^2)}{(\lambda^2 + \gamma_m^2)\gamma_m}$$
(4.5)

$$\varphi_m = \frac{2}{L} \frac{(\beta \lambda^2 + 1)(w_0^{\prime\prime} - w_L^{\prime\prime}(-1)^m)}{(\lambda^2 + \gamma_m^2)} \quad ; \qquad \bar{\varphi}_0 = \frac{\beta}{L} (w_0^{\prime\prime} - w_L^{\prime\prime}) + \frac{w_0 - w_L}{L}$$
(4.6)

By considering Eq. (3.8)-(3.12), the second derivatives of the lateral displacement can be obtained as follows:

$$\frac{d^2 w}{dx^2} = -\frac{H^*}{D_x} (M_x + M_x^T) \quad ; \quad H^* = \frac{G^*}{G^* + N_x}$$
(4.7)

This means that the second derivatives of the lateral displacement at both ends are associated with the in-plane moments in the x-direction to derive any arbitrary boundary conditions.

#### 4.1. Boundary conditions

In order to employ Stokes' transformation in any combination of the boundary conditions, the beam is assumed to be elastically restrained by setting the rotational ( $K_2$  and  $K_4$ ) and translational ( $K_1$  and  $K_3$ ) springs at the two ends, as shown in Fig. 2. The geometrical and natural boundary conditions at x=0 are presented as follows:

$$\begin{cases} M_x^0|_{x=0} = -K_2\varphi_0\\ Q_{xz}^0 + N_x \frac{dw}{dx}\Big|_{x=0} = -K_1w_0 \end{cases}$$
(4.8)

 $M_x^0$  and  $Q_{xz}^0$  are moment and shear force per length at x=0, respectively. In the same manner, the boundary conditions are presented at x=L and written similar to relation (4.8). Consequently, four equations are extracted in terms of  $w_0$ ,  $w_L$ ,  $\varphi_0$  and  $\varphi_L$ . By applying Stokes' transformation technique to relation (4.8), the natural and geometrical boundary conditions can be rewritten in terms of end values as follows:

$$K_{2} \left[ \frac{\beta H^{*}}{LD_{\chi}} (M_{\chi}^{L} - M_{\chi}^{0}) + \frac{w_{0} - w_{L}}{L} \right] + K_{2} \left[ \frac{2H^{*}}{LD_{\chi}} (\beta \lambda^{2} + 1) \sum_{m=1}^{\infty} \left( \frac{M_{\chi}^{L}(-1)^{m} - M_{\chi}^{0} + M_{\chi}^{T}((-1)^{m} - 1)}{\lambda^{2} + \gamma_{m}^{2}} \right) \right] + M_{\chi}^{0} = 0$$

$$(4.9)$$

$$\begin{split} & (G^* + N_x) \left[ \frac{w_L - w_0}{L} + \frac{2H^*}{LD_x} \sum_{m=1}^{\infty} \left( \frac{M_x^0 - M_x^L (-1)^m + M_x^T (1 - (-1)^m)}{\lambda^2 + \gamma_m^2} \right) \right] + \\ & G^* \left[ \frac{\beta H^*}{LD_x} \left( M_x^L - M_x^0 \right) + \frac{w_0 - w_L}{L} \right] + G^* \left[ \frac{2H^*}{LD_x} \left( \beta \lambda^2 + 1 \right) \sum_{m=1}^{\infty} \left( \frac{M_x^L (-1)^m - M_x^0 + M_x^T ((-1)^m - 1)}{\lambda^2 + \gamma_m^2} \right) \right] K_1 w_0 = 0 \end{split}$$

Similar equations can be derived for x=L, leading to obtaining the four linear algebraic equations containing four unknown boundary values as follows:

$$[A_{ij}]\{w_0, w_L, M_x^0, M_x^L\}^I = 0 \quad ; \quad (i, j = 1, 2, 3, 4)$$
(4.11)

A non-trivial solution implies that the determinant of the coefficients of the matrix  $[A_{ij}]$  is vanished, which results in an equation whose lowest root is the critical buckling temperature. The components of the matrix  $[A_{ij}]$  are given in Appendix B.

4.1.1. Rigid boundary conditions

#### 4.1.1.1. Simply-simply supported beam

When the beam is simply supported at the two ends (x=0, x=L), the values of translational and rotational spring stiffness coefficients are presented as follows:

$$x = 0: Simply Supported \Rightarrow \begin{cases} K_1 \to \infty \\ K_2 \to 0 \end{cases}; \quad x = L$$
  
$$: Simply \ Supported \Rightarrow \begin{cases} K_3 \to \infty \\ K_4 \to 0 \end{cases}$$
(4.12)

By substituting the above-mentioned values in characteristic equation of matrix  $\left[A_{ij}\right]\!\!,$  we can obtain

$$G^* - d_{11}T - \frac{G^*}{G^* + D_x \gamma_m^2} = 0 \quad ; \quad d_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\alpha(z)dz \tag{4.13}$$

#### 4.1.1.2. Clamped-clamped beam

In this case, the translational and rotational spring stiffness approaches infinity in both ends, and the boundary conditions for clamped-clamped beam are achieved as follows:

$$x = 0: Clamped \Rightarrow \begin{cases} K_1 \to \infty \\ K_2 \to \infty \end{cases} ; \quad x = L: Clamped \Rightarrow \begin{cases} K_3 \to \infty \\ K_4 \to \infty \end{cases}$$
(4.14)

By putting the above-mentioned values in characteristic equation of matrix  $\left[A_{ij}\right]\!\!,$  we can obtain

$$\sum_{m=1}^{\infty} \left( \frac{1 - (-1)^m}{\lambda^2 + \gamma_m^2} \right) = 0 \tag{4.15}$$

#### 4.1.1.3. Clamped-simply supported beam

In this case, the values of translational and rotational spring stiffness coefficients are presented as follows:

$$x = 0: Clamped \Rightarrow \begin{cases} K_1 \to \infty \\ K_2 \to \infty \end{cases}; \quad x = L: Simply Supported$$
$$\Rightarrow \begin{cases} K_3 \to \infty \\ K_4 \to 0 \end{cases}$$
(4.16)

In a similar manner, by replacing the above-mentioned values in the characteristic equation of matrix  $[A_{ij}]$ , the following form is obtained:

$$\sum_{m=1}^{\infty} \left(\frac{1}{\lambda^2 + \gamma_m^2}\right) = -\frac{\beta}{2(\beta\lambda^2 + 1)}$$
(4.17)

#### 4.1.2. Elastically restrained beam

It is supposed that at x=0 the beam is clamped and restrained at x=L by translational and rotational ( $K_3$  and  $K_4$ ) springs:

$$x = 0 \Rightarrow \begin{cases} K_1 = 10^6 \\ K_2 = 10^6 \end{cases} ; \qquad x = L \Rightarrow \begin{cases} 0 < K_3 < \infty \\ 0 < K_4 < \infty \end{cases}$$
(4.18)

By substituting the above-mentioned values in the characteristic equation of matrix  $[A_{ij}]$ , the following equation is obtained to determine the critical buckling temperature:

$$\frac{2H^{*}}{LE^{*}}(S_{m} - H_{m})\left[\frac{A_{1}}{L} + \frac{2H^{*}}{LE^{*}}\beta\frac{K_{4}}{L}(A_{1} + G^{*})(\lambda^{4}(S_{m} + H_{m})(\beta + 1) + \beta\lambda^{2} + 1) + \frac{2H^{*}}{LE^{*}}K_{3}K_{4}\left((S_{m} + H_{m})(1 + \beta^{2}\lambda^{4}) + (1 + 2\lambda^{2}(S_{m} + H_{m}))\right)\right] -$$

$$\frac{2H^{*}}{LE^{*}}\left(K_{3}H_{m}(\beta\lambda^{2} - 1) - \frac{\beta K_{3}}{2}\right) = 0$$
(4.19)

When both sides of the beam are supported by translational and rotational boundary conditions as follows, we have:

$$x = 0 \Rightarrow \begin{cases} 0 < K_1 < \infty \\ 0 < K_2 < \infty \end{cases}; \qquad \qquad x = L \Rightarrow \begin{cases} 0 < K_3 < \infty \\ 0 < K_4 < \infty \end{cases}$$
(4.20)

#### 5. Results

Using Matlab software, we developed a computer program to ascertain the accuracy and reliability of Stokes' transformation technique with the available solution. To this end, some numerical solutions are presented to investigate the influence of material and geometrical parameters on critical temperature value in buckling. Functionally-graded materials (FGMs) with a mixture of silicon and nitride as the ceramic (Si3N4) and stainless steel (SUS304) as the metal are utilized to examine the present method. Material properties, including Young's modulus and thermal expansion coefficient, are shown in Table 1. Poisson's ratio is v = 0.3 for metal and ceramic constituents.

Material	E (Gpa)	α (1/°C)
Si <sub>3</sub> N <sub>4</sub>	322.2	7.475e-6
SUS304	207.79	15.321e-6

Table 1: Material properties

In all numerical results, a shear correction factor of  $\pi^2/12$  is employed [43]. It is assumed that the beam is exposed to uniform temperature rise across the thickness. Variation in modulus of elasticity and thermal expansion coefficient for two different distributions of ceramic and metal is FGM1<sub>(a=1/b=1/c=2/p)</sub> and FGM2<sub>(a=1/b=0/c/p)</sub>, respectively. The illustrations of Young modulus variations along with the graphs and details are available in Fig. 3. As observed, the FG material distribution obeys a symmetric profile in FGM1, whereas in FGM2 the material distribution through the thickness has an asymmetric pattern. For the sake of brevity, the details are not presented here; the details are available in [48].



Fig. 3. Variations of the Young modulus through the thickness for cases (a) FGM1 and (b) FGM2

#### 5.1. Rigid boundary conditions

Since infinite Fourier series is applied as the solution method in the present study, evaluating the convergence of the results is necessary. For this purpose, the critical value of temperature for a homogenous beam (p=0) with two types of end conditions (both ends are clamped (CC) and clamped-simply supported (CS)) and the slenderness ratio L/h=25 are calculated in Table (2), using different numbers of Fourier series terms. The results are compared with those reported in [10]. As shown in Table 2, a good convergence is observed when 55 terms of the

series are utilized. In table 3, convergence study is carried out for the beam when is supported on clamped edge at x=0 whereas the values of the translational and rotational stiffness of the springs at the other end vary from  $10^2$  to  $10^6$ . As listed, 40 terms truncated from the infinite Fourier series give an acceptable accuracy of the critical buckling temperature for the beam with elastically-restrained ends.

Boundary Conditions		Number of polynomial terms of finite series in calculating thermal buckling (FGM2)								
		10	20	30	40	50	55	60	65	
0	CC	763.101	722.198	718.511	710.907	703.341	698.951	698.951	698.951	
p = 0	CS	420.396	389.150	368.451	362.200	360.847	358.977	358.970	358.970	
10	CC	751.622	710.719	707.032	699.428	691.862	687.472	687.381	687.381	
p = 10	CS	418.595	387.349	366.650	360.399	359.046	357.176	357.17	357.17	

Table 2: Convergence in the present method in case of L/h=25

Table 3: Convergence in the present method for a beam with the clamped, elastically restrained ends

(K3,K4)		Number of polynomial terms of finite series							
		10 20		30	40	50			
n-0	(10 <sup>2</sup> ,10 <sup>2</sup> )	354.8153	353.9487	346.656	344.509	344.5080			
p=o	(106,106)	353.0566	358.2737	353.009	352.876	352.8740			
p=1	(10 <sup>2</sup> ,10 <sup>2</sup> )	235.5653	230.6987	226.4060	224.2588	224.2580			
	(106,106)	259.9667	259.1838	258.9190	258.7860	258.7840			

Thermal buckling analysis of two models of FGM1 and FGM2 under different rigid boundary conditions for various FGM power indices and slenderness ratios (L/h) are presented in Tables 4 and 5, respectively. As observed, the critical temperature values for FGM1 model get more than those for FGM2 model. Moreover, for the two models of FGMs, the critical temperatures for CC boundary condition attain the highest value. The results of thermal buckling for FGM2 are compared with those obtained based on the closed-form solution in [10]. It is seen that there is a very good agreement between the results, confirming the high accuracy of the current methodology. Finally, it should be noted that the Young modulus of Silicon Nitride is greater than that of stainless steel. Therefore, for p=0, the beam exhibits the homogenous characteristics value of stiffness; on the other hand, increasing the FGM power index does not cause the ceramic volume fraction to decrease the beam stiffness significantly. Consequently, for all values of slenderness ratio in Table 5, the greatest value of critical temperature corresponds to p=0, and the critical temperature decreases along with the increase in the FGM power index.

Boundary	I/h	n						
Conditions	L/II	Ψ						
		0	1	3	10			
	10	4067.8	3524.5	3046.9	2503.7			
	25	698.95	605.59	523.53	430.20			
CC	30	485.29	426.14	363.83	300.51			
	35	358.17	314.06	286.19	221.55			
	40	275.79	240.90	206.54	169.98			
	10	2156.4	2028	1808	1660.5			
	25	358.97	337.30	301.14	276.76			
CS	30	249.87	234.78	209.63	192.67			
	35	183.84	172.73	154.24	141.76			
	40	140.88	132.37	118.21	108.64			
	10	1080.5	943.95	808.98	664.90			
	25	176.31	154.10	132.16	108.70			
SS	30	122.58	107.14	91.89	75.58			
	35	90.12	78.77	67.56	55.57			
	40	69.03	30.34	51.75	42.57			

Table 4: The variation of the critical value of the FG beam temperature for  $FGM1_{(a=1/b=1/c=2/p)}$ 

Boundary	I/b	/h n=0 n=1 n=2		n-2		n-10				
Conditions	L/II	<i>p=0</i>		<i>p=1</i>	<i>p</i> =1		<i>p=3</i>		<i>p=10</i>	
-	-	Present	Ref.	Presen	Ref.	Presen	Ref.	Presen	Ref.	
		work	[10]	t work	[10]	t work	[10]	t work	[10]	
	10	4067.8	3992.2	2687.7	2646.7	2403.5	2353.3	2192.3	2159.5	
	25	698.95	692.81	461.82	458.76	412.98	409.41	376.70	376.20	
СС	30	485.29	483.50	322.78	320.13	288.33	285.76	264.02	262.61	
	35	358.17	356.29	237.92	235.89	211.90	210.60	195.22	193.54	
	40	275.79	273.31	181.70	180.95	163.01	161.56	149.26	148.48	
	10	2156.4	2128.4	1428.6	1410.2	1273.1	1256.2	1169.2	1153.5	
	25	358.97	356.88	237.67	236.29	212.19	210.94	195.01	193.86	
CS	30	249.84	248.53	165.43	164.54	147.71	146.91	135.76	135.02	
	35	183.84	182.90	121.71	121.09	108.68	108.12	99.89	99.37	
	40	180.88	140.19	93.27	92.81	83.29	82.87	76.55	76.17	
	10	1080.5	1072.9	715.52	710.53	638.34	633.83	586.50	582.33	
	25	176.31	175.34	116.72	116.08	104.23	103.65	95.79	95.27	
SS	30	122.58	121.91	81.15	80.71	72.47	72.07	66.61	66.24	
	35	90.12	89.64	59.66	59.34	53.28	52.99	48.97	48.71	
	40	69.03	68.66	45.70	45.46	40.81	40.59	37.51	37.31	

Table 5: Comparison of critical temperature in this study and Ref. [10] for FGM2 (a=1/b=0/c/p)

### 5.2. Elastically-restrained boundary conditions

In this section, thermal buckling of a beam with flexible edge conditions is investigated. In Fig. 4, variation in the buckling temperature for different values of L/h and p for the two models of FGM1/2 are sketched with respect to variation of FG power index and  $K_3$  and  $K_4$ . At x=0, the values of the translational and rotational stiffness of springs are considered to be fixed ( $K_1=K_2=10^{10}$ ). As seen in Fig. 4, the critical buckling temperature rises more for FGM1 in comparison with FGM2; by increasing ( $K_3$ ,  $K_4$ ) = ( $10^5$ ,  $10^4$ ) to ( $K_3$ ,  $K_4$ ) = ( $10^7$ ,  $10^6$ ), the buckling temperature reaches to its maximum value.





Fig. 4. Variation of the critical buckling temperature  $(T_{cr})$  for different values of  $(K_3, K_4)$ , (L/h), and p in the two models of FGM1/2

Fig. 5 demonstrates the variation of the critical value of temperature when the translational value of elastic support is considered to be fixed as ( $K_1=K_3=10^{10}$ ). The influences of different values of rotational stiffness factors ( $K_2$  and  $K_4$ ), different FGM power indices, and slenderness ratio on the responses of the beam in thermal buckling for the two models of FGM1/2 are presented. As seen in Fig. 5, the critical buckling temperature rises more for FGM1 in comparison with FGM2; by increasing ( $K_2$ ,  $K_4$ ) = ( $10^5$ ,  $10^4$ ) to ( $K_3$ ,  $K_4$ ) = ( $10^7$ ,  $10^6$ ), the buckling temperature reaches to its maximum value.





Fig. 5. Variations of the critical buckling temperature ( $T_{cr}$ ) for different values of ( $K_2$ ,  $K_4$ ), (L/h), and p in the two models of FGM1/2

# 6. Conclusion

In the present study, the thermal buckling behavior of FG beam was studied, using the trigonometric form of Fourier series expansion and Stokes' transformation technique. To that end, the governing equations in buckling based on Von-Karman nonlinearity with Timoshenko beam assumption were derived. Furthermore, the influence of the four-parameter power-law FG model on critical temperature was considered. It was found that the proposed method can handle both elastically-restrained and rigid edge supports. The results can be summarized as follows:

- Increasing the slenderness ratio enhances the critical buckling temperature, whereas increasing FG power index decreases it. These are all due to the fact that the beam behavior gets closer to the pure metallic material.
- As observed, the critical temperature values for FGM1 model get more than those for FGM2 model. Moreover, for the two models of FGMs, the critical temperatures for CC boundary condition attain the highest value.
- Increasing the translational and rotational spring coefficient makes the beam stiffer and prevents it from being flexible when exposed to the thermal variation of the environment; as a result, the critical buckling temperature is increased.
- The influence of two models of metal-ceramic distribution across the thickness(FGM1/2 models) on the responses of the beam in thermal buckling of FG

beam is shown that the critical buckling temperature rises more for symmetrical model of FGM beam with respected to unsymmetrical one.

- For all values of slenderness ratio, the greatest value of critical temperature corresponds to homogenous ceramic beam (p=0) with clamped-clamped boundary condition, whereas the critical temperature decreases along with the increase in the FGM power index (p>0).
- The proposed approach may be extended to study thermal buckling as well as postbuckling of beams, plates, and shells when subjected to arbitrary boundary conditions.

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#### Appendix A

#### Lateral displacement derivatives in Stokes' transformation technique

By defining the lateral displacement in sine Fourier series in the interval of 0<x<L in the

following form, we have:

$$w(x) = \sum_{m=1}^{\infty} w_m \sin\left(\frac{m\pi x}{L}\right) \quad 0 < x < L \tag{8.1}$$

The first order and the higher order of derivatives of w(x) are obtained by applying Stokes' transformation technique; the boundary values are separately determined without using relation (8.1) in order to involve them in finding the derivatives of w(x) containing the unknown boundary values at the two ends of the beam [38]:

$$\frac{dw}{dx} = \frac{w_L - w_0}{L} + \sum_{m=1}^{\infty} \left[ \frac{2}{L} (w_L (-1)^m - w_0) + \gamma_m w_m \right] \cos(\gamma_m x) \qquad 0 \le x \le L$$
(8.2)

$$\frac{d^2 w}{dx^2} = -\sum_{m=1}^{\infty} \gamma_m \left[ \frac{2}{L} (w_L (-1)^m - w_0) + \gamma_m w_m \right] \sin(\gamma_m x) \qquad 0 < x < L \quad (8.3)$$

$$\frac{d^{3}w}{dx^{3}} = \frac{w_{L}^{\prime\prime} - w_{0}^{\prime\prime}}{L} + \sum_{m=1}^{\infty} \left[\frac{2}{L} (w_{L}^{\prime\prime} (-1)^{m} - w_{0}^{\prime\prime}) - w_{m}^{2} \left(\frac{2}{L} \{w_{L} (-1)^{m} - w_{0}^{\prime\prime} - w_{0}^{\prime\prime}$$

$$\frac{d^4 w}{dx^4} = -\sum_{m=1}^{\infty} \gamma_m \left[ \frac{2}{L} (w_0^{\prime\prime} - w_L^{\prime\prime} (-1)^m) + \gamma_m^2 \cdot \left( \frac{2}{L} \{ w_L (-1)^m - w_0 \} + 0 < x < L \right]$$

$$0 < x < L \quad (8.5)$$

$$(8.5)$$

#### Appendix **B**

The components of coefficients matrix  $[A_{ij}]$ 

$$\begin{split} A_{11} &= \frac{K_2}{L} ; A_{12} = -\frac{K_2}{L} ; A_{13} = -\frac{K_2 H^*}{L D_x} \left[ \beta + 2(\beta \lambda^2 + 1) \sum_{m=1}^{\infty} \left( \frac{1}{\lambda^2 + \gamma_m^2} \right) \right] + 1 ; \\ A_{14} &= \frac{K_2 H^*}{L D_x} \left[ \beta + 2(\beta \lambda^2 + 1) \sum_{m=1}^{\infty} \left( \frac{(-1)^m}{\lambda^2 + \gamma_m^2} \right) \right] \\ A_{21} &= \frac{K_4}{L} ; A_{22} = -\frac{K_4}{L} ; A_{23} = -\frac{K_4 H^*}{L D_x} \left[ \beta + 2(\beta \lambda^2 + 1) \sum_{m=1}^{\infty} \left( \frac{(-1)^m}{\lambda^2 + \gamma_m^2} \right) \right] \\ A_{24} &= \frac{K_4 H^*}{L D_x} \left[ \beta + 2(\beta \lambda^2 + 1) \sum_{m=1}^{\infty} \left( \frac{1}{\lambda^2 + \gamma_m^2} \right) \right] + 1 \end{split}$$

$$\begin{aligned} A_{31} &= -\frac{A_1}{L} + K_1 ; A_{32} = \frac{A_1}{L} ; A_{33} = \frac{H^*}{L D_x} \left[ 2(A_1 - G^* \beta \lambda^2) \sum_{m=1}^{\infty} \left( \frac{1}{\lambda^2 + \gamma_m^2} \right) - \beta G^* \right] \\ A_{41} &= -\frac{A_1}{L} ; A_{42} = \frac{A_1}{L} + K_3 ; A_{43} = \frac{H^*}{L D_x} \left[ 2(A_1 - G^* \beta \lambda^2) \sum_{m=1}^{\infty} \left( \frac{(-1)^m}{\lambda^2 + \gamma_m^2} \right) - \beta G^* \right] \end{aligned}$$

$$A_{44} = -\frac{H^{*}}{LD_{x}} \left[ 2(A_{1} - G^{*}\beta\lambda^{2}) \sum_{m=1}^{\infty} \left(\frac{1}{\lambda^{2} + \gamma_{m}^{2}}\right) - \beta G^{*} \right]$$

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