

Seismic performance optimization of reinforced concrete shells generated by the force density method: A nonlinear multilayered approach

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Article Info	Abstract
Article History: Received 04 Aug 2026 Accepted 01 Oct 2026 Keywords: Concrete shells; Geometric optimization; Non-linear analysis; Time-history analysis; Form-finding; Earthquake design; Buckling; Stiffness degradation; Force density method; Nonlinear multi-layered model; Geometric imperfections; Seismic reintroduction	The construction of concrete shell roofs in Mexico has declined owing to seismic vulnerability and slenderness effects. Traditional form-finding guarantees static equilibrium under self-weight alone and remains inadequate under seismic demand, where stiffness degradation and stress reversal govern the response. Three form-found anticlastic shells with four, six and eight supports were generated by the force density method, which yields compression-dominant configurations before reinforcement is defined. Reinforcement was first designed with the Eurocode 2 three-layer sandwich model, which neglects the compressive contribution of the reinforcement; a nonlinear multilayered formulation was then adopted to resolve the stress state through the shell thickness. Design actions followed NTC-2023; the analyses incorporated initial geometric imperfections, nonlinear buckling, the Takeda hysteretic model and synthetic accelerograms for Mexico City. The shells developed controlled, low-level nonlinear behavior, with compression governing the response and the reinforcement carrying a significant fraction of the compressive demand under stress reversal. Eigenmode-affine imperfections reduced the ultimate capacity by 45.2% to 22.1%, the reduction diminishing with the number of supports: more petals stiffen the shell and raise its capacity, at the cost of ductility. Since imperfections consume so large a share of that capacity, recovering design margin becomes decisive: accounting for the compressive contribution of the reinforcement reduces the required area by 10%, making the multilayered representation a design requirement rather than a refinement. The numerical cracking patterns coincide spatially with the damage documented after the 2017 earthquake in Félix Candela's Los Manantiales restaurant, supporting a framework transferable to other earthquake-prone regions.

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1. Introduction

Over the past eighty years, concrete shell structures have played a fundamental role in the development of industrial activities, primarily serving as containment systems for materials. Their main advantage lies in their ability to cover large areas with high span-to-thickness ratio, efficiently distributing stress while requiring very small quantities of concrete and reinforcing steel. Concrete shell roofs depart from traditional rectilinear forms of conventional construction. Today, they are widely employed in contemporary architecture for iconic global projects where designers develop

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progressively more complex and challenging free-form structures with high aesthetic and sculptural value. This evolution is the result of significant advances in computational modeling, geometric optimization and numerical analysis techniques. Recent research has extended form-finding beyond the purely static domain: [1] implemented the natural force density method within an efficient parametric computational framework, and [2] established the conditions for the existence and uniqueness of force density solutions, together with the singularities that constrain the method. In parallel, the seismic behavior of thin concrete shells has received renewed attention: [3] assessed the performance of Félix Candela's shells during the 1985 Mexico City earthquake, [4] quantified damage and stiffness degradation in a doubly curved concrete shell under spatially variable seismic excitation, and [5] characterized the influence of initial geometric imperfections on the buckling capacity of single and double curvature concrete shells.

In Mexico, the construction of concrete shell structures experienced a period of significant expansion between 1960 and 1980, driven by the work of renowned architects such as Fernando López Carmona, Pedro Ramírez Vázquez, and Félix Candela [6,7]. During that period, even national construction codes from the 1980s included specific sections dedicated to the design and verification of shell concrete structures. However, over time and given that Mexico is located in a high-seismicity zone, many of these shells suffered significant irreversible damage during seismic events, leading to costly repairs and, in some cases, demolition due to their inability to comply with modern safety standards. Consequently, although interest in shell structures among architects and engineers remains high, their construction has declined substantially. The main reasons are the inadequacy of traditional methods in predicting nonlinear behavior and stiffness degradation under seismic loads; stability, buckling, and cracking issues inherent to slender shells, as well as time-dependent effects such as creep and the overall complexity of construction processes.

The objective of this article is to apply contemporary geometric optimization to the nonlinear time-history analysis of a set of form-found anticlastic concrete shells subjected to the static and dynamic load conditions established by the Mexican construction codes (NTC 2023) [8], incorporating geometric nonlinearity and initial geometric imperfections. Among the available form-finding strategies, the force density method was selected because it transforms the nonlinear equilibrium problem into a single system of linear equations, avoiding the iterative convergence difficulties inherent to dynamic relaxation and transient stiffness approaches[9], and providing both computational efficiency and numerical stability in the definition of the reference geometry a decisive advantage when that geometry must subsequently serve as the starting configuration for nonlinear seismic analyses. The goal is to propose an analytical methodology for the design of concrete shell structures that contributes to the development of technical guidelines and design criteria. Although the methodology is calibrated here against the seismic hazard and soil conditions of Mexico City, it constitutes a general framework applicable to thin-shell structures in any region of high seismic risk.

The originality of this work does not lie in applying existing optimization techniques to a standard seismic analysis. Its contribution is the identification and resolution of a specific limitation of current design practice: the three-layer sandwich model prescribed by Eurocode 2 (EN 1992-2, Informative Annex LL) [10] assumes that the reinforcement resists tension exclusively and assigns the entire compressive capacity to the concrete core. Under bilateral seismic loading with stress reversal, that assumption does not represent the actual state of the section, and it leads systematically to conservative and uneconomical shells. The present study replaces it with a nonlinear multilayered formulation that resolves the stress state through the shell thickness and explicitly captures the compressive contribution of the reinforcement, and quantifies, through nonlinear time-history analyses, both the magnitude of the discrepancy and the design efficiency that can be recovered.

2. Concrete Shells and Form Finding

A shell is defined as a three-dimensional curved structure whose thickness is significantly smaller than its other two dimensions. Its primary advantage lies in its ability to transfer external loads

into internal membrane stresses of tension and compression, while neglecting bending and shear stresses. Depending on their geometry, shells can be classified into [11]:

- Free-form shells: Geometries generated without any simple mathematical constraints, primarily designed according to aesthetic principles. They can be digitally described using high-degree complex polynomials, such as NURBS surfaces.
- Mathematical shells: Structures generated from analytical mathematical functions. These forms were the main types used for concrete shell construction due to their practicality in closed-form solution calculations and their symmetry, which facilitated construction.
- Form-finding shells: Geometries whose primary purpose is to achieve a static equilibrium state of the structure. This category includes scaled physical models as well as advanced computational simulation models.

Among these three categories, the present study focuses on form-found shells. The reason is structural rather than aesthetic. In seismic regions, a form-found geometry is one configured to carry its loads predominantly through membrane action, minimizing the parasitic bending moments that arise when the surface departs from a funicular configuration. A moment-free or nearly moment-free stress state maximizes the exploitation of the compressive capacity of concrete, which is its strongest response, and reduces the tensile demand that governs cracking and reinforcement requirements. This has a direct dynamic consequence: the high geometric stiffness and low mass resulting from such configurations place the fundamental frequencies of the shell above the dominant frequencies of seismic excitation, a mechanism identified for small and mid-sized shells in [12]. Geometry is therefore treated here as the first line of seismic design, defined before any reinforcement is considered.

2.1. Form-Finding Methods

From their origins, concrete shells have been remarkable for their aesthetic value, often resembling natural forms, while requiring rigorous analysis and calculations to perform safely and efficiently under both permanent and accidental loads. The origins of shell-type structures can be traced back to the 18th and 19th centuries with the construction of iconic buildings worldwide, including churches, auditoriums, temples, and theaters.



Fig. 1. The idea behind La Sagrada Familia: Gaudí's hanging chain model. Available from: <https://parametric-architecture.com/la-sagrada-familia-gaudis-chain-model/>

The period of significant expansion of shell structures began primarily in Europe, especially in Italy, Germany, and Spain. In Spain, the experimental development of the “form-finding” concept was advanced by architect Antonio Gaudí, who employed analog scale models consisting of suspended chains and threads under gravity, generating catenaries and inverted arches subjected to tensile forces. These models had then materialized into architectural constructions that relied primarily on compressive forces, enhancing structural stability without sacrificing sculptural and aesthetic

value. Continuing with research on slender shell structures and innovations in materials and construction techniques, it is essential to highlight the work of German architect Frei Otto, who introduced the concepts of “minimal surface” and “minimal work” using soap films and spider webs to create experimental analog models of geometric optimization. These studies promoted structural efficiency and material optimization in the geometric design of shell and tensile structures, laying theoretical and practical foundations for spatial, branched, pneumatic, and other advanced structural systems.

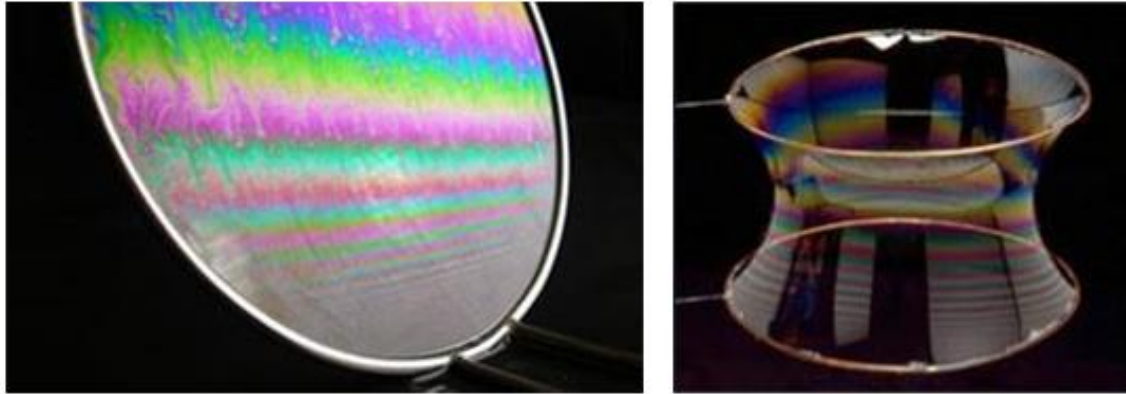


Fig. 2. Soap films, minimal surfaces and beyond, Hong Kong Laureate Forum. Available from: <https://www.hklaureateforum.org/en/soap-films-minimal-surfaces-and-beyond>

Over time, with the advent of computer-aided design (CAD) and finite element numerical analysis from the 1950s to the present, increasingly complex free-form structures have been modeled, analyzed, and understood using optimization and parametrization techniques based on algorithms and geometric rules. These methods allow arbitrary surfaces to be numerically optimized and described under predefined loading conditions, typically considering self-weight only, while accounting for the following parameters [13]:

- Model topology.
- Boundary conditions and supports.
- External loads applied.
- Relationship between internal stress and geometry.

By satisfying these conditions, the geometric design of shell structures becomes an efficient and controlled iterative parametric process. Principal methods employed in this context include the Transient Stiffness Method, the Force Density Method, and the Dynamic Relaxation Method [9].

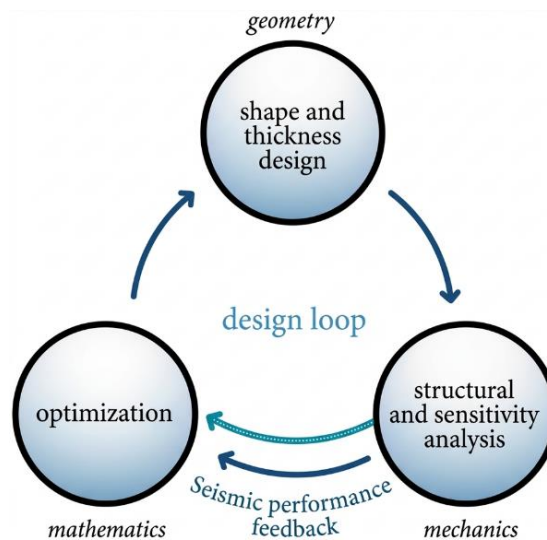


Fig. 3. Steps of form-finding optimization [13]

2.2. Force Density Method

The Force Density Method enables the determination of the equilibrium geometry of a series of points connected through a cable network or a compression structure, where the loads are discretized into nodal point loads. Based on the formulations of K. Linkwitz and H.J. Schek [14,15] in the 1970s, this optimization aims to minimize the potential strain energy of a surface by satisfying static equilibrium through stress redistributions, solving the algebraic equilibrium systems using linear approximations, which makes the calculations simple, efficient, and robust.

The approach applied to concrete shells is based on the creation of a grid model that emulates membrane behavior. This grid achieves its static equilibrium configuration under the influence of external loads acting at discrete joints. Subsequently, these concentrated actions are transformed into distributed loads along the linear elements. It is imperative that the following conditions are satisfied for the grid system to achieve equilibrium in three-dimensional space:

$$s_a \cos(a, x) + s_b \cos(b, x) + s_c \cos(c, x) + s_d \cos(d, x) = p_x \quad (1)$$

$$\text{direction cos} = \frac{\text{coordinate difference}}{\text{distance in space}} \quad (2)$$

$$\frac{s_a}{a}(x_m - x_i) + \frac{s_b}{b}(x_j - x_i) + \frac{s_c}{c}(x_k - x_i) + \frac{s_d}{d}(x_l - x_i) = p_x \quad (3)$$

By incorporating the force density concept also referred to as tension coefficients into the equations to address the nonlinear characteristics of the equilibrium problem, it can be expressed three-dimensionally as:

$$\text{force density} = \frac{\text{force in a bar}}{\text{stressed length of the bar}} \quad (4)$$

$$q_a(x_m - x_i) + q_b(x_j - x_i) + q_c(x_k - x_i) + q_d(x_l - x_i) = p_x \quad (5)$$

$$q_a(y_m - y_i) + q_b(y_j - y_i) + q_c(y_k - y_i) + q_d(y_l - y_i) = p_y \quad (6)$$

$$q_a(z_m - z_i) + q_b(z_j - z_i) + q_c(z_k - z_i) + q_d(z_l - z_i) = p_z \quad (7)$$

The analog models of Gaudí and Frei Otto and the force density method are not merely historically related: they address the same equilibrium problem in two different representations. A soap film carries a uniform, isotropic surface tension and therefore adopts the configuration of minimum area compatible with its boundary, a minimal surface of zero mean curvature. In the force density method, the ratio $q = s/l$ between the force in a branch and its length plays the role of that uniform tension, and the linear system of Eqs. (5) to (7) is the stationarity condition of the functional $\sum q l^2/2$ under the prescribed nodal loads and boundary conditions. Assigning a constant force density to every branch therefore minimizes the sum of squared branch lengths: the network converges to the discrete surface of least Dirichlet energy, which coincides with the minimal surface when the branch lengths are uniform and closely approximates it otherwise. The exact correspondence with the soap-film geometry is recovered by updating the force densities in inverse proportion to the branch lengths, as formalized in the natural force density method [1].

In this study a uniform force density is assigned within each model. The departure from the minimal surface is therefore introduced by the applied load and by the prescribed crown height, not by the force density field. This departure modifies the rise and the curvature of the shell and, consequently, its dynamic properties, which is the mechanism exploited here for seismic purposes (Section 3).

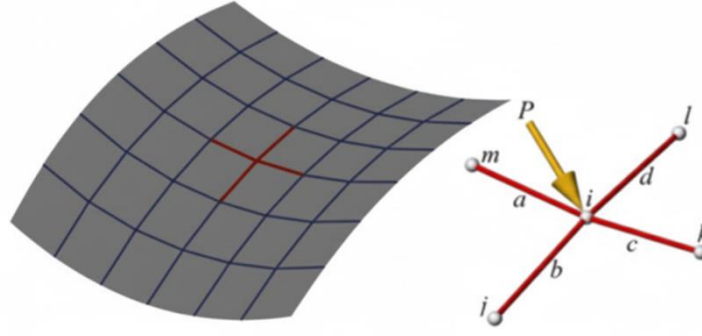


Fig. 4. Generic mesh and discretization of elements and loads at a generic node using Rhinoceros 3D- Grasshopper software

2.2.1. Matrix Formulations

Following the determination of the general equilibrium solution for an individual three-dimensional node, the mathematical framework must be extended to determine the static equilibrium of surfaces characterized by arbitrary geometries and an indeterminate number of free and fixed nodes. This problem is addressed through the synergistic application of matrix algebra and graph-theoretic principles.

The methodology relies on the application of branch-node incidence matrices of size $(m \times n)$, with m branches and n nodes) to define the system's topology and the concurrent evaluation of partial derivatives via the Jacobian matrix. Within this approach, the connectivity between the nodes and branches of the system is formally described by a matrix, denoted as C , which is governed by the following principles:

$$C_{ij} = \begin{cases} +1 = & \text{if branch } j \text{ ends in node } i \\ -1 = & \text{if branch } j \text{ begins in node } i \\ 0 = & \text{otherwise} \end{cases} \quad (8)$$

The matrix C is strictly a branch-node incidence matrix that captures only the topological relationship of the mesh, being completely independent of the system's geometric or metric properties.

$$\begin{matrix} i & j & k & l & m \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} & \begin{matrix} a \\ b \\ c \\ d \end{matrix} \end{matrix} \quad (9)$$

The branch-node incidence matrix, C , is subsequently partitioned into two constituent submatrices: one corresponding to the unconstrained (free) nodes and the other to the constrained (fixed) nodes (representing known boundary conditions). Concurrently, the vector of nodal coordinates is similarly subdivided according to the same partitioning scheme.

$$C = [C_N \quad C_F] \quad (10)$$

$$x = [x_N \quad x_F] \quad (11)$$

By resolving the problem along the x , y , and z directions, the following expressions are obtained:

$$u = C_N x_N + C_F x_F \quad (12)$$

$$v = C_N y_N + C_F y_F \quad (13)$$

$$w = C_N z_N + C_F z_F \quad (14)$$

By defining U , V , and W as the diagonal matrices corresponding to u , v , and w , respectively, along with the lengths of the bars in the mesh, the following formulation can be established:

$$L = (U^2 + V^2 + W^2)^{\frac{1}{2}} \quad (15)$$

By formulating the equilibrium equations of the Force Density Method in matrix form within Euclidean space and incorporating the transpose of the Jacobian operator, the following results are obtained:

$$\left(\frac{\partial f(x_i)}{\partial x_i} \right)^T = \begin{bmatrix} \frac{\partial l_a}{\partial x_i} & \frac{\partial l_b}{\partial x_i} & \frac{\partial l_c}{\partial x_i} & \frac{\partial l_d}{\partial x_i} \end{bmatrix} = \begin{bmatrix} \frac{-(x_j - x_i)}{l_a} \\ \frac{-(x_k - x_i)}{l_b} \\ \frac{-(x_l - x_i)}{l_c} \\ \frac{-(x_m - x_i)}{l_d} \end{bmatrix}^T \quad (16)$$

$$\left(\frac{\partial f(x)}{\partial x} \right)^T = C_N^T U L^{-1} \quad (17)$$

Through the incorporation of the force vectors $\mathbf{f}$ and the load component vector $\mathbf{p}$, along with the force density in each bar $Uq = QCx$, the objective is to derive the system of linear equations in the following form:

$$C_N^T Q C_N x_N + C_N^T Q C_F x_F + p = 0 \quad (18)$$

$$D_N = C_N^T Q C_N \quad (19)$$

$$D_F = C_N^T Q C_F \quad (20)$$

In this formulation, the problem is represented as a system of linear equations in the form $\mathbf{Ax}=\mathbf{b}$ where the three-dimensional static equilibrium can be expressed for as:

$$x_N = D_N^{-1}(p_X - D_F x_F) \quad (21)$$

$$y_N = D_N^{-1}(p_Y - D_F y_F) \quad (22)$$

$$z_N = D_N^{-1}(p_Z - D_F z_F) \quad (23)$$

The coordinate vectors obtained from Eqs. (21) to (23) define the reference configuration of the shell. This configuration is not merely a geometric output: it is the geometry on which the finite element discretization is built, and therefore the geometry that determines both the linear stiffness matrix K_L and the geometric stiffness matrix K_G of Eq. (57), which governs the nonlinear response. The partition of the incidence matrix introduced in Eq. (10) is what carries the boundary conditions into that configuration: the constrained nodes x_F are the supports of the shell, and their number and position condition the modal properties, the buckling capacity of the resulting structure, as shown in Sections 5.1.2., 5.2 and 5.5. The force density method is therefore not only a form-finding tool but the first stage of the seismic analysis itself.

3. Concrete Shell Roof Models

The design of the concrete shell structures was conducted using visual programming within Rhinoceros 3D, employing the Grasshopper plugin and implementing the Force Density Method [1] on a quadrilateral mesh anchored at the base of the structures. The resulting surfaces are anticlastic, i.e., of negative Gaussian curvature, and their petals resemble hyperbolic paraboloids; however, they are form-found rather than analytical hypars, and the term anticlastic is used throughout this paper to denote them.

The form-finding network is a radial quadrilateral net whose supports and central crown are fixed nodes, with the crown held at a prescribed height. A unit load is applied at every free node, and a single force density q is assigned to every branch, both quantities being dimensionless. This uniform assignment is deliberate. It leaves the plan layout of the network determined solely by topology and boundary conditions, so that q scales only the load induced elevation. It yields branch forces proportional to branch lengths, i.e. the most uniform funicular membrane state, and a symmetric positive-definite system with a unique, singularity free solution [2]. Finally, it defines each geometry by a single scalar, which keeps the comparison between models reproducible.

With supports and crown fixed, q is the only shape parameter, and its admissible range is bounded on both sides. Low values make the load-induced component dominant: the free edges rise excessively, and the crown becomes a local depression. High values flatten the petals, increasing their radius of curvature and reducing buckling capacity, while the horizontal thrust at the supports grows in direct proportion to q . The adopted values (0.50, 0.60 and 0.70) were selected within this range, and their adequacy was verified a posteriori through the modal and nonlinear buckling analyses of Section 5. Since q depends on the size of each network, the values are model-specific and are not compared across models.

The fixed crown is a boundary condition of the form-finding only; in the finite element models it is a free node. Any residual force carried by the crown in the network is therefore not available in the built shell and must be resisted locally by bending. This is consistent with the concentration of the cracking index around the crown (Section 5.4.1). It is also the reason why the structural response is assessed with a full nonlinear shell model rather than inferred from the form-finding equilibrium.

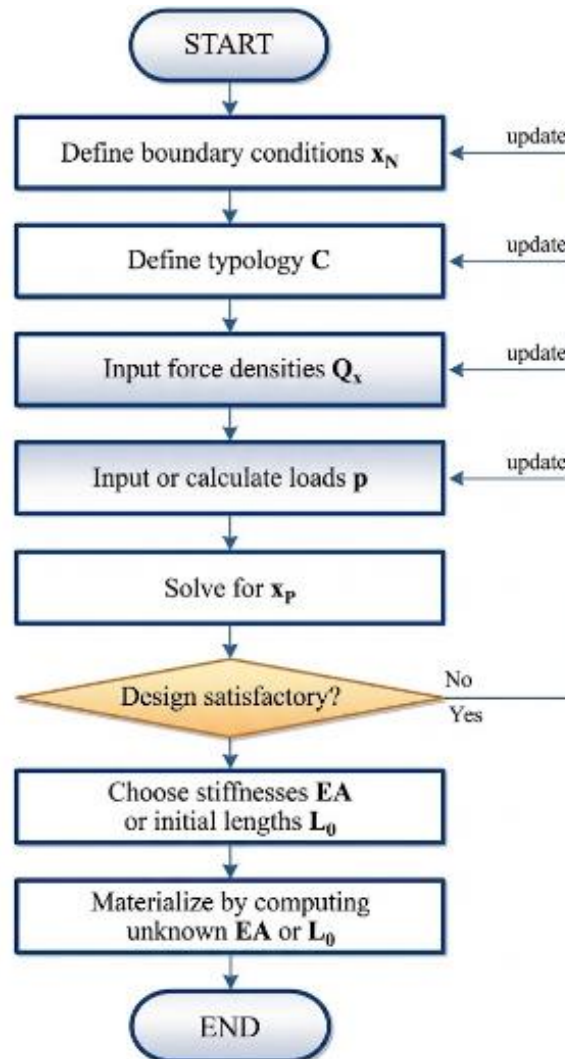


Fig. 5. Force density method flowchart [11]

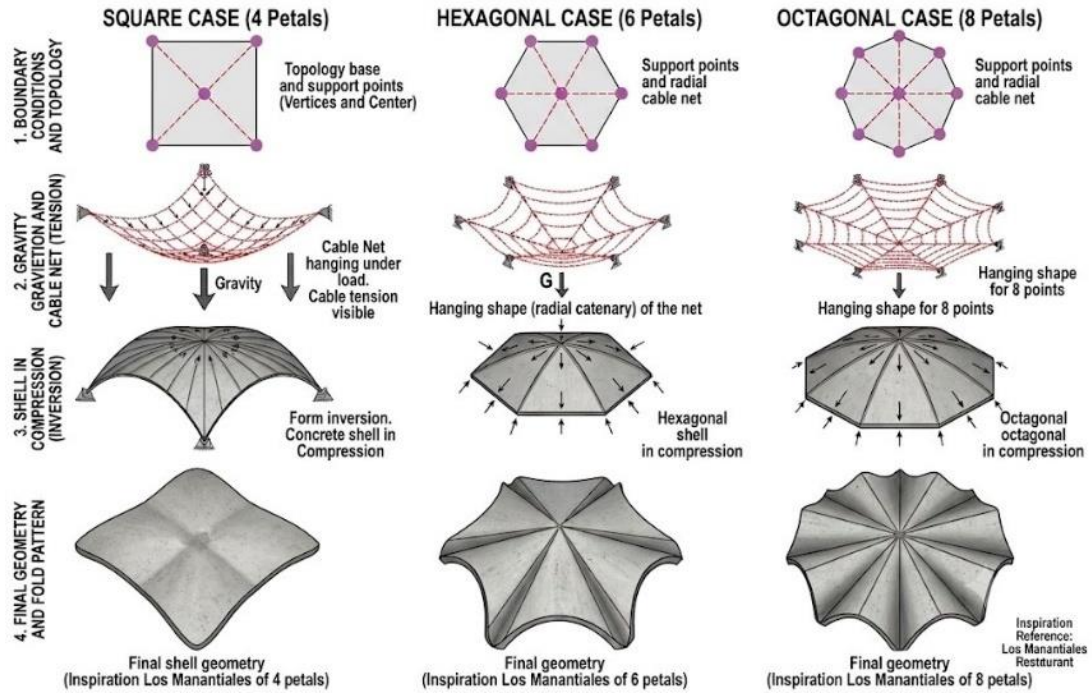


Fig. 6. Analysis model generation process using the force density method

3.1. Geometrical Characterization of The Models

The geometric and discretization parameters of the three models are summarized in Table 1. All shells share a thickness of 18 cm and differ in the number of perimeter supports and in the reference force density assigned during form-finding. Measured from the support plane, the rise-to-span ratios are 0.13, 0.15 and 0.15 for PH-04, PH-06 and PH-08, respectively, within the range usually associated with shallow shells. No shallow shell simplification is adopted: all analyses use the full three-dimensional geometry with a geometrically nonlinear formulation. Table 1 reports, for each model, the force density, the size of the form-finding network, the crown height above the supports (3.76, 5.32 and 4.74 m) and the max. height above the supports (5.16, 6.89 and 6.13 m).

Table 1. Geometric, form-finding and finite element properties of the shell models

Property	Unit	PH-04	PH-06	PH-08
Number of supports	–	4	6	8
Plan dimensions	m	39.90 × 39.90	44.97 × 42.22	39.60 × 39.60
Span between adjacent supports	m	25.25	20.03	13.45
Maximum height above supports ^a	m	5.16	6.89	6.13
Rise-to-span ratio ^a	–	0.13	0.15	0.15
Shell thickness	mm	180	180	180
Force density, q ^b	–	0.50	0.60	0.70
Crown height above supports ^c	m	3.76	5.32	4.74
Element type	–	Layered, Quad 4	Layered, Quad 4	Layered, Quad 4
Boundary conditions	–	Fixed	Fixed	Fixed
Target element size	mm	500	450	450
Nodes / elements ^d	–	2,197 / 1,941	4,881 / 4,400	6,169 / 5,872

^a Measured from the support plane; rise-to-span ratio = maximum height / maximum plan dimension; ^b Uniform in every branch of the form-finding network, with a unit load at every free node (dimensionless); ^c Prescribed in the form-finding only; the crown is a free node in the finite element models; ^d Analysis mesh before automatic meshing.

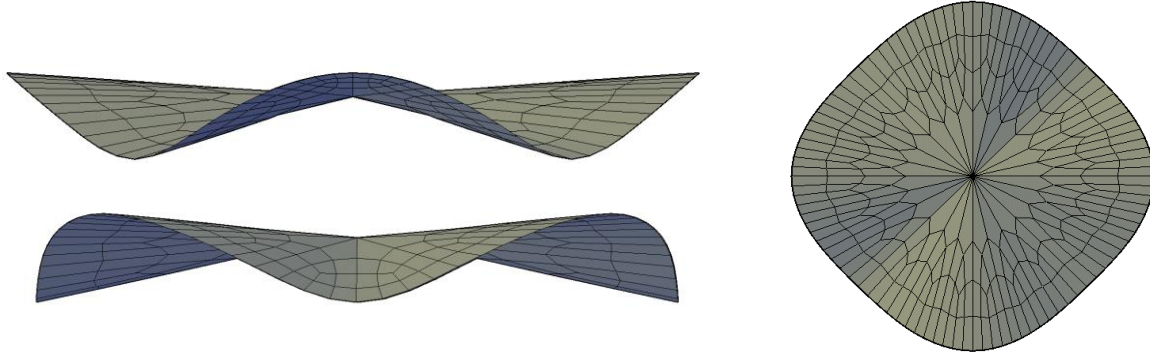


Fig. 7. Model PH-04: four anticlastic petals, uniform force density $q = 0.50$ (dimensionless, unit nodal load), crown fixed 3.76 m above the support plane

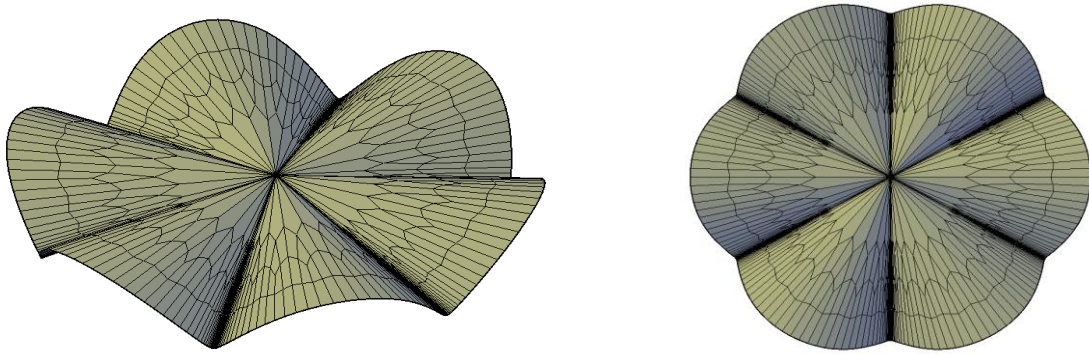


Fig. 8. Model PH-06: six anticlastic petals, uniform force density $q = 0.60$ (dimensionless, unit nodal load), crown fixed 5.32 m above the support plane

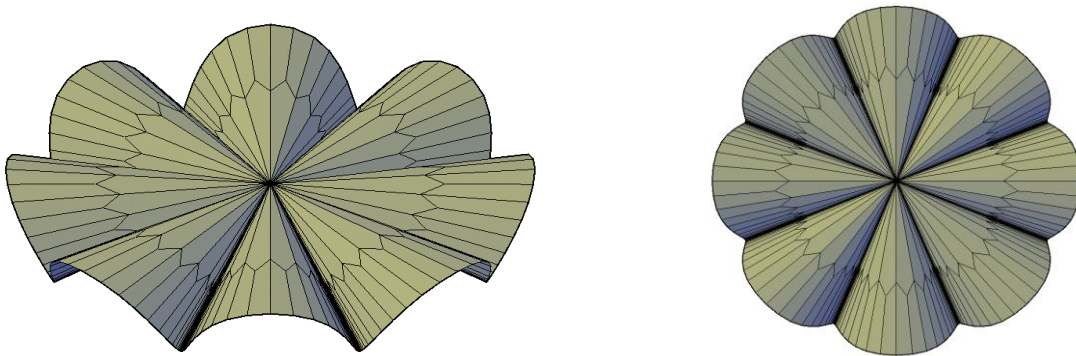


Fig. 9. Model PH-08: eight anticlastic petals, uniform force density $q = 0.70$ (dimensionless, unit nodal load), crown fixed 4.74 m above the support plane

3.2. Nonlinear Models for Concrete and Steel

The importance of accounting for the constitutive behavior of nonlinear materials stems from the need to accurately predict the structural response and the onset of damage when loads surpass the elastic limits of the material. Understanding this behavior allows engineers to capture inelastic deformations, cracking, and other forms of structural deterioration that occur beyond the linear-elastic range, which is critical for safe and efficient design. Moreover, incorporating the nonlinear properties is essential to assess the enhanced structural capacity provided by reinforcing steel, as its contribution becomes significant once the concrete begins to yield or crack.

The nonlinear model proposed by Mander et al. [16] is adopted for concrete. It reproduces strain hardening, post-peak softening and, in particular, the effect of confinement on the stress-strain response. This last feature is decisive in the sections considered here: in a 180 mm shell the

reinforcement layers occupy a substantial proportion of the thickness, so that the degree of confinement provided to the concrete core directly governs the descending branch of the response and the deformation capacity of the section. An elasto-plastic constitutive model [17] is adopted for reinforcement, reproducing the initial elastic response, yielding and plastic deformation; it is compatible with the layered formulation described in Section 5.5.2. It should be noted that the compressive softening reproduced by the material model operates at the constitutive level: the cracking criterion of EN 1992-2 Annex LL adopted in Section 4.2 for the sectional verification does not evaluate concrete crushing, a distinction discussed in Section 6.

Table 2. Properties of the concrete employed in the analyses

Compressive strength (MPa)	Tensile strength (MPa)	Modulus of elasticity (MPa)	Shear modulus (MPa)	Poisson 's ratio	Weight per unit volume (kN/m ³)
24.52	3.08	21,707	9,044.79	0.20	23.54

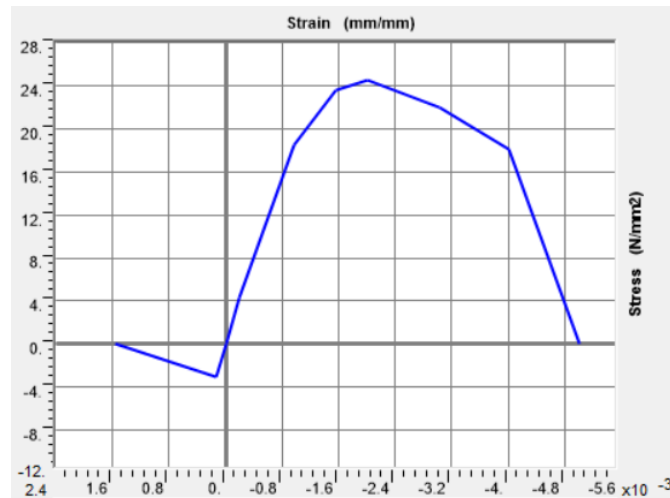


Fig. 10. Constitutive model of the nonlinear behavior of concrete in analyses (1 Mpa=1N/mm²)

Table 3. Properties of the steel in rebars employed in the analyses

Minimum Yield Stress (MPa)	Minimum tensile stress (MPa)	Modulus of elasticity (MPa)	Shear modulus (MPa)	Poisson 's ratio	Weight per unit volume (kN/m ³)
413.68	620.53	199,948	76,903.07	0.30	76.88

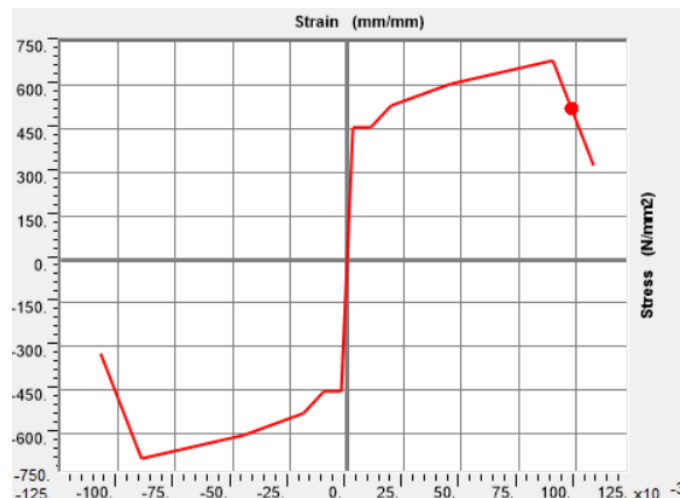


Fig. 11. Constitutive model of the nonlinear behavior of steel in analyses (1 Mpa=1N/mm²)

3.3. Imperfection Theory in Concrete Shells

As noted above, the efficiency of concrete shells lies in their ability to cover large spans with a high span-to-thickness ratio, a performance that depends critically on their geometry [11]. This same efficiency, however, makes them extremely sensitive to deviations from the idealized analysis model. Construction processes are subject to measurable and controllable imperfections, so that variations in formwork, material quality and element thickness may reduce the load-carrying capacity of the structure below the idealized theoretical values assumed in design [5,18,19]. In thin concrete shells these perturbations trigger geometric instability (buckling) and bending stresses that are not anticipated when the analysis relies solely on membrane theory, which in turn causes premature cracking of the concrete, stress redistribution, geometric and material nonlinear effects, and time-dependent effects such as creep [20]. The seismic damage sustained by Félix Candela's concrete shells in Mexico City during the 1985 earthquake, documented in the assessment of the Church of Our Lady of the Miraculous Medal [3], illustrates the need to account for such deviations from the structural design stage onwards.

3.3.1. Koiter's Formulations

The formal study of geometric imperfections in thin shells originates in the post-buckling theory published by Koiter in 1945 [21], which introduced the concept of imperfection sensitivity. Koiter demonstrated that the sensitivity of the buckling load to initial imperfections is governed by the structure of the expansion of the total potential energy about the critical equilibrium state, written as a Taylor series in the amplitude of the buckling modes, Eq. (24):

$$V = V_0 + \frac{1}{2} V_{ii} \xi_i^2 + \frac{1}{6} V_{ijk} \xi_i \xi_j \xi_k + \frac{1}{24} V_{ijkl} \xi_i \xi_j \xi_k \xi_l + \dots \quad (24)$$

Where V_{ijk} denotes the imperfection amplitude parameter V_{ijkl} and the post-buckling coefficient.

Defining a reduction factor that relates the actual critical loads (P_{crit}) affected by imperfections, to the theoretical critical loads (P_{ideal}) according to Eq. (25):

$$(1 - \rho) \rho^2 = \beta \cdot \delta \quad (25)$$

Where ρ is the load reduction, δ the dimensionless imperfection amplitude and β the imperfection-sensitivity exponent.

Within the framework of geometric imperfection theory, the critical thickness is obtained by adopting Zoelly's formulation for the critical buckling pressure [22], Eq. (26):

$$p_{cr} = \frac{2E}{\sqrt{3(1 - \nu^2)}} \left(\frac{t}{R} \right)^2 \quad (26)$$

Where E is the modulus of elasticity (Young's modulus) of the material, ν the Poisson's ratio of the material, t the shell thickness and R the radius of curvature of the shell.

Equation (27) is accordingly proposed as a preliminary design expression for the minimum thickness required to ensure the stability of concrete shells under design loads in the presence of geometric imperfections:

$$t \geq \sqrt{\frac{P_{design} \cdot R}{f'_c \cdot \lambda}} \quad (27)$$

Where P_{design} denotes the applied load and λ the reduction factor incorporating Koiter's imperfection-sensitivity parameter β .

The buckling capacity of thin shells is known to be extremely sensitive to deviations from the design geometry, but the magnitude of that sensitivity is not a general property of shells: it depends on the geometry considered. The classical imperfection-sensitivity formulations, and the knock-down factors derived from them, were established for cylindrical shells under axial compression and for spherical shells under external pressure, for which experimental capacities as low as 20% of the

classical analytical prediction have been reported [23,24]. In both geometries the critical load is associated with many buckling modes clustered within a narrow range of load factors, and it is this near simultaneity of modes, through the modal interaction it induces, that produces such severe reductions. Doubly curved shells of negative Gaussian curvature behave differently: their critical mode is generally well separated from the subsequent ones, modal interaction is correspondingly weaker, and the reductions in critical load reported for these geometries are substantially smaller [5]. Transferring a knock-down factor calibrated for cylindrical or spherical shells to an anticlastic surface is therefore not justified and would lead to an unnecessarily conservative design. Initial geometric imperfections nevertheless remain one of the variables with the greatest influence on the limit states of thin concrete shells [18,20], and characterizing them for the specific geometry analyzed is what allows the thickness of a concrete shell to be calibrated against the execution tolerances of its construction process rather than against idealized theoretical values. For this reason, the knock-down factor adopted in this study is not taken from the literature but determined for the shells analyzed here, through the imperfection-sensitivity series reported in Section 5.1.3.

4. Reinforced Concrete Shell Design

The primary objective of an optimal design for reinforced concrete shell structures is to determine the required amount of reinforcing steel within their thickness to resist normal stresses, bending moments, and shear forces. Due to the thin-shell configuration of these structures, it is essential to evaluate potential cracking caused by tensile stresses. While the previously employed geometric optimization models are intended to ensure that the overall behavior of the shells is primarily compressive, their limitation lies in considering only self-weight effects. This implies that, although the static equilibrium is satisfied under compression, it does not fully represent the range of loading conditions that these structures will experience throughout their service life, which includes service loads, seismic actions, time-dependent effects, and geometric nonlinearities where tensile stresses may arise.

Consequently, the design of the employed models was conducted following a combination of the updated 2023 Mexican National Building Codes [8,25], from which the applied loads and load combinations for a linear static analysis phase were obtained, yielding the design envelopes for the previously described analysis models. Subsequently, Eurocode 2: Design of Concrete Structures Part 1-1: General Rules and Rules for Buildings (EN 1992-1-1, 2004) and Eurocode 2: Design of Concrete Structures Part 2: Concrete Bridges Design and Detailing Rules (EN 1992-2, 2005) [10,26] were applied to determine the required steel reinforcement ratios for layer reinforcement in the shell elements (three layered design model). The designs obtained from this combined regulatory approach were further evaluated through a nonlinear time-history analysis, using synthetic accelerograms developed for Mexico City. This assessment aimed to validate the structural efficiency of geometric optimization and form-finding models and to demonstrate the significant reduction of tensile stresses in anticlastic shell structures, thereby confirming that their construction in high seismic risk areas can be carried out safely.

The design procedure adopted in this study constitutes a hybrid regulatory approach, and it is declared as such. Seismic hazard characterization, load combinations and displacement verification follow the Mexican Complementary Technical Standards [8], whereas the layer-by-layer reinforcement design follows Eurocode 2 [10,26]. This combination was adopted because the Mexican standards do not provide an explicit procedure for the design of shell elements subjected to the eight stress resultants, whereas Informative Annex LL of EN 1992-2 does. The two frameworks are combined at a level at which they remain consistent: the Mexican standards supply the demand, through site-specific spectra and load combinations, while the European standards supply the sectional resistance model. Partial safety factors are applied within each framework and are not mixed within a single verification. The effect of this choice is confined to the reinforcement design stage: the nonlinear time-history analyses reported in Section 5 are governed by the constitutive models, the geometry and the ground motions, and are therefore independent of the code used to size the reinforcement, which enters only through the reinforcement layout adopted in the layered sections.

4.1. Design Load Combinations Per The 2023 Mexican Building Code (NTC)

The NTC-2023 stipulates that, to verify the safety of a structure, the effects of various actions (loads) must be considered in combination. For seismic design, the fundamental combinations include permanent actions (Dead Load), variable actions (Live Load), and accidental actions (Seismic Load).

Load Combination for Gravitational Effects:

$$P = 1.3 \text{ Dead Load} + 1.5 \text{ Live Load} \quad (28)$$

4.1.1. Response Spectrum for Mexico City

The elastic design spectrum for a specific site in Mexico City, considering the seismic zone and soil type, defines the maximum spectral acceleration and the plateau period, which are critical for modal dynamic analysis. This spectrum represents the seismic forces per unit weight of the structure and is adjusted by the seismic behavior factor in the final design.

Using the SASID application, the response spectra for the analyses were generated, along with synthetic accelerograms for a region in Mexico City, which were subsequently employed in the nonlinear time-history analyses.

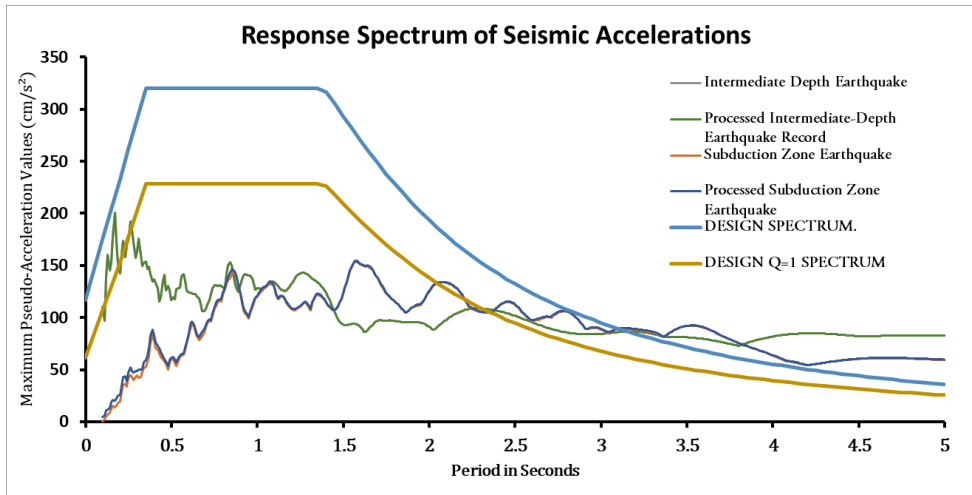


Fig. 12. Elastic design spectrum based on the Mexico City Building Code (NTC-2023), generated with SASID software [27]

A design seismic response spectrum is proposed for Mexico City at a site with coordinates 19.4023243, -99.1889867. The analyzed shell structures are assumed to be located on firm ground to avoid differential settlements and potential geotechnical or foundation-related issues that could affect structural performance.

4.1.2. Design Load Envelope in Accordance with Mexican Regulations

The combination of dead loads and seismic actions is defined as follows:

$$P1 = 1.1 \text{ Dead Load} + 1.1 \text{ Live Load} + (Sx + 0.30 Sy) \quad (29)$$

$$P2 = 1.1 \text{ Dead Load} + 1.1 \text{ Live Load} + (Sx - 0.30 Sy) \quad (30)$$

$$P3 = 1.1 \text{ Dead Load} + 1.1 \text{ Live Load} + (Sy + 0.30 Sx) \quad (31)$$

$$P4 = 1.1 \text{ Dead Load} + 1.1 \text{ Live Load} + (Sy - 0.30 Sx) \quad (32)$$

With Sx and Sy denoting the seismic forces along the x and y axes, respectively.

4.2. Cracking Evaluation of Shell-Type Structures

According to the Eurocode provisions, this type of structure is evaluated for cracking at three different locations along its cross section (upper, middle, and lower zones) as follows:

$$\text{Top} = \begin{bmatrix} S_{11}^t & S_{12}^t & 0 \\ S_{21}^t & S_{22}^t & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} \sigma_1^t \\ \sigma_2^t \\ \sigma_3^t = 0 \end{cases} \quad (33)$$

$$\text{Center} = \begin{bmatrix} S_{11}^c & S_{12}^c & S_{13}^c \\ S_{21}^c & S_{22}^c & S_{23}^c \\ S_{31}^c & S_{32}^c & 0 \end{bmatrix} \rightarrow \begin{cases} \sigma_1^c \\ \sigma_2^c \\ \sigma_3^c \end{cases} \quad (34)$$

$$\text{Bottom} = \begin{bmatrix} S_{11}^b & S_{12}^b & 0 \\ S_{21}^b & S_{22}^b & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} \sigma_1^b \\ \sigma_2^b \\ \sigma_3^b = 0 \end{cases} \quad (35)$$

The shear stresses at the midsection are calculated using the Jourawski method [28], according to:

$$S_{13}^c = 1.5 \frac{V1}{h} \quad (36)$$

$$S_{23}^c = 1.5 \frac{V2}{h} \quad (37)$$

Where $V1$ and $V2$ are the resultant shear forces in their respective planar directions.

For the assessment and design of shell-type concrete structures, the function Φ defines a multiaxial strength criterion for concrete, essentially representing a failure surface. Its origin lies in advanced plasticity/fracture models (such as the Drucker-Prager and modified Rankine criteria), calibrated experimentally.

The function employs stress invariants $I1$ (mean stress) and $J2$ (shear stress) to assess whether a point is subjected to critical stress levels. If $\Phi \leq 0$, the concrete remains uncracked; if $\Phi > 0$, it fails or cracks. This criterion is primarily used in nonlinear finite element analyses (FEA) and is not part of the direct design formulas in Eurocode 2 2005 Annex LL

$$\Phi = \alpha \frac{J_2}{f_{cd}^2} + \lambda \frac{\sqrt{J_2}}{f_{cd}} + \beta \frac{I_1}{f_{cd}} - 1 \begin{cases} \leq 0 & \text{Uncracked} \\ > 0 & \text{Cracked} \end{cases} \quad (38)$$

$$J_2 = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad (39)$$

$$J_3 = (\sigma_1 - \sigma_m)(\sigma_2 - \sigma_m)(\sigma_3 - \sigma_m) \quad (40)$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \quad (41)$$

$$\sigma_m = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) \quad (42)$$

For reasons of brevity, only the main formulations of the concrete cracking assessment method proposed by European standards are presented in this article. For a complete review of the equations and the methodology employed in this research, the reader is referred to Informative Annex LL, "Concrete Shell Elements," of Eurocode 1992-2:2005 [10].

It should be noted that the application of this method exclusively considers phenomena associated with concrete cracking, thereby neglecting the effects of concrete crushing. As a result, the design

of shell-type elements is limited to tensile reinforcement bars, while compressive steel reinforcement is disregarded. Under accidental seismic loads with bilateral loading and unloading cycles, this simplification does not fully reflect the structural response.

Consequently, in this study, shell-type structures are designed using an optimization-based form-finding model, promoting structural performance primarily under compressive stresses. This approach assumes that the concrete core of the cross section absorbs compressive forces, while the reinforcing steel counteracts tensile stresses (Concrete Three-Layered Model Design).

It is well established that, under accidental loads, internal stress combinations occur within the cross section (tension in some fibers and compression in others). Accordingly, this article subsequently presents a transparent nonlinear time-history and buckling analysis under a nonlinear multi-layered model. The aim is to evaluate the contribution of each material within the cross section, assess the necessity of compressive steel reinforcement, and verify the accuracy and efficiency of current design methods, including the validity of their simplifying assumptions.

4.3. Three-Layer Design Model for Reinforced Concrete Shells

Shell elements are subjected to eight stress resultants: three membrane forces (N_{11} , N_{22} , N_{12}), two bending moments (M_{11} , M_{22}), a twisting moment (M_{12}), and two transverse shear forces (V_1 , V_2). For design, the shell is idealized as a sandwich model with two outer layers carrying membrane and bending forces, and an uncracked core carrying transverse shear [29,30].

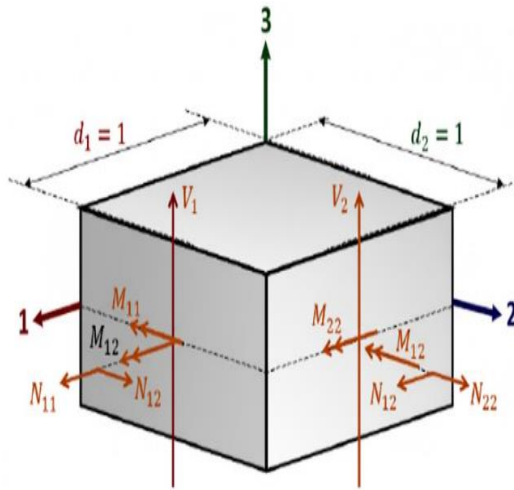


Fig. 13. Unit shell element and stress resultants

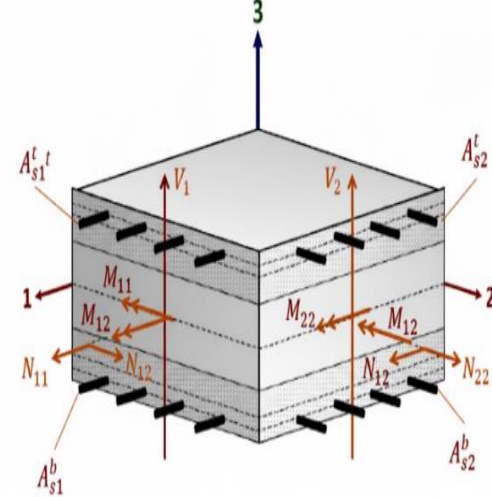


Fig. 14. Idealized three layered model

The membrane stresses associated with the upper and lower regions of the cross section of the shell-type structure, as well as the forces required to resist the reinforcing steel (top and bottom) and the concrete in the central region, are defined as follows:

$$\sum F_1 = 0 \rightarrow N_{s11}^t = N_{11}^t + N_{12}^t \tan \theta \quad (43)$$

$$\sum F_2 = 0 \rightarrow N_{s22}^t = N_{22}^t + N_{12}^t \cot \theta \quad (44)$$

$$\sum F_1 = 0 \rightarrow N_c^t = -\frac{N_{12}^t}{\sin \theta \cos \theta} \quad (45)$$

The moments associated with the membrane stresses at the mid-plane of the layers are established according to:

$$\sum M_{\text{center of the bottom layer}} = 0 \rightarrow N_{11}^t = \frac{N_{11} h^b - M_{11}}{h^t + h^b} \quad (46)$$

$$\sum M_{center\ of\ the\ bottom\ layer} = 0 \rightarrow N_{22}^t = \frac{N_{22}h^b - M_{22}}{h^t + h^b} \quad (47)$$

$$\sum M_{center\ of\ the\ bottom\ layer} = 0 \rightarrow N_{12}^t = \frac{N_{12}h^b - M_{12}}{h^t + h^b} \quad (48)$$

The same formulas are assumed for the upper layer.

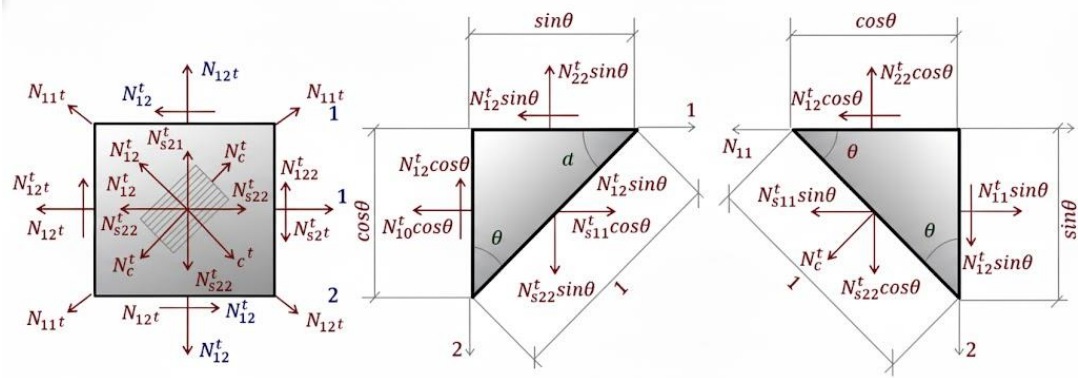


Fig. 15. Top layer forces(left); section parallel to the crack (center); section perpendicular to the crack (right)

For the case of this study, the reinforcing steel in the concrete shells is assumed to be present in both layers, upper and lower. Consequently, the forces that must be resisted by the reinforcing steel in both layers are defined as follows:

$$N_{s11}^t = N_{11}^t + |N_{12}^t| \cot \theta \quad (49)$$

$$N_{s22}^t = N_{22}^t + |N_{12}^t| \tan \theta \quad (50)$$

$$N_c^t = \frac{|N_{12}^t|}{\sin \theta \cos \theta} \quad (51)$$

The angle θ denotes the orientation of the principal compressive stress field within the concrete with respect to the reference axis of the element.

The stresses in the reinforcing steel are calculated when such reinforcement is required in both layers, according to:

$$\sum M_{bottom\ reinforcement} = 0 \rightarrow N_{d11}^t = \frac{N_{s11}^t(h^t + h_{s1}^b) + N_{s11}^b(h_{s1}^t - h^b)}{(h_{s1}^t + h_{s1}^b)} \quad (52)$$

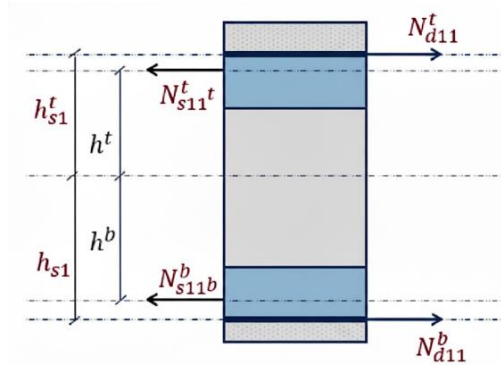


Fig. 16. Loads on the reinforcement for both layers in sandwich element

The required reinforcement area per unit length and the design thicknesses of the bottom and top layers are determined as:

$$A_{s1}^t = \frac{N_{d11}^t}{f_{yd}} \quad (53)$$

$$A_{s2}^t = \frac{N_{d22}^t}{f_{yd}} \quad (54)$$

$$a^t = \frac{N_c^t}{f_c} \quad (55)$$

The same calculations are performed for the lower layer of the cross section. Here f_{yd} is the design yield strength of the reinforcement and f_{cd} the design compressive strength of the concrete, both derived from the characteristic values reported in Table 2 and 3 through the partial safety factors $\gamma_s = 1.15$ and $\gamma_c = 1.5$ prescribed by [10,26]. Only the main formulations applicable to the models of this study are presented; the reader is referred to [31] for a comprehensive treatment.

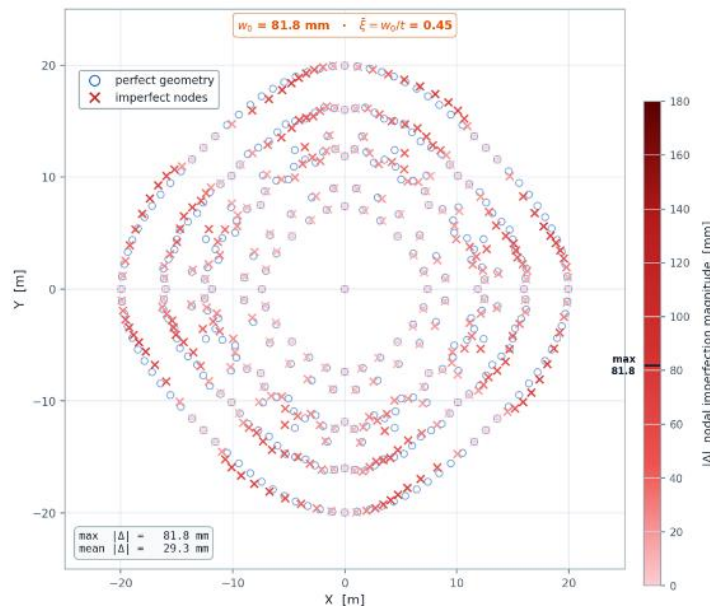
5. Nonlinear Numerical Analysis of Shell-Type Concrete Models

All results reported in this section correspond to the imperfect configuration of the three shells. The perfect geometry is retained solely as a reference for the fundamental periods and the buckling capacity, in order to quantify the loss of stiffness and of load-carrying capacity introduced by the geometric imperfections.

5.1. Initial Geometric Imperfections and Imperfection Sensitivity

5.1.1. Generation Of the Imperfect Geometry

The formulations of Section 3.3.1 are applied here to the three shells. For each model the first eigenmode of a linear buckling analysis under gravity loading was extracted, normalized and superimposed on the nodal coordinates [32] according to Eq. (24). Following Koiter's principle, the mode-affine field is the most degrading imperfection shape when the critical mode is isolated, so that the configurations analyzed represent the most unfavorable case for the amplitude adopted. Since the mode vanishes at the supports, the boundary conditions of the perfect and imperfect models are identical, and the comparison isolates the effect of the geometric deviation alone. Figures 17 to 19 show both geometries; Section 5.1.2 quantifies the resulting loss of capacity.



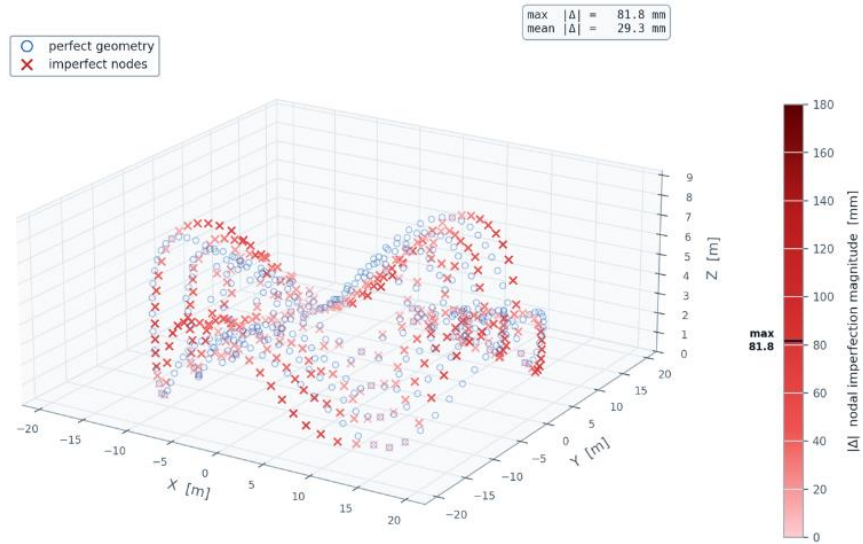


Fig. 17. Perfect and imperfect nodal geometries of the PH-04 model for $w_0 = 81.8$ mm ($\xi = 0.45$, dimensionless): (a) plan view, (b) 3D view; color: $|\Delta|$ (mm)

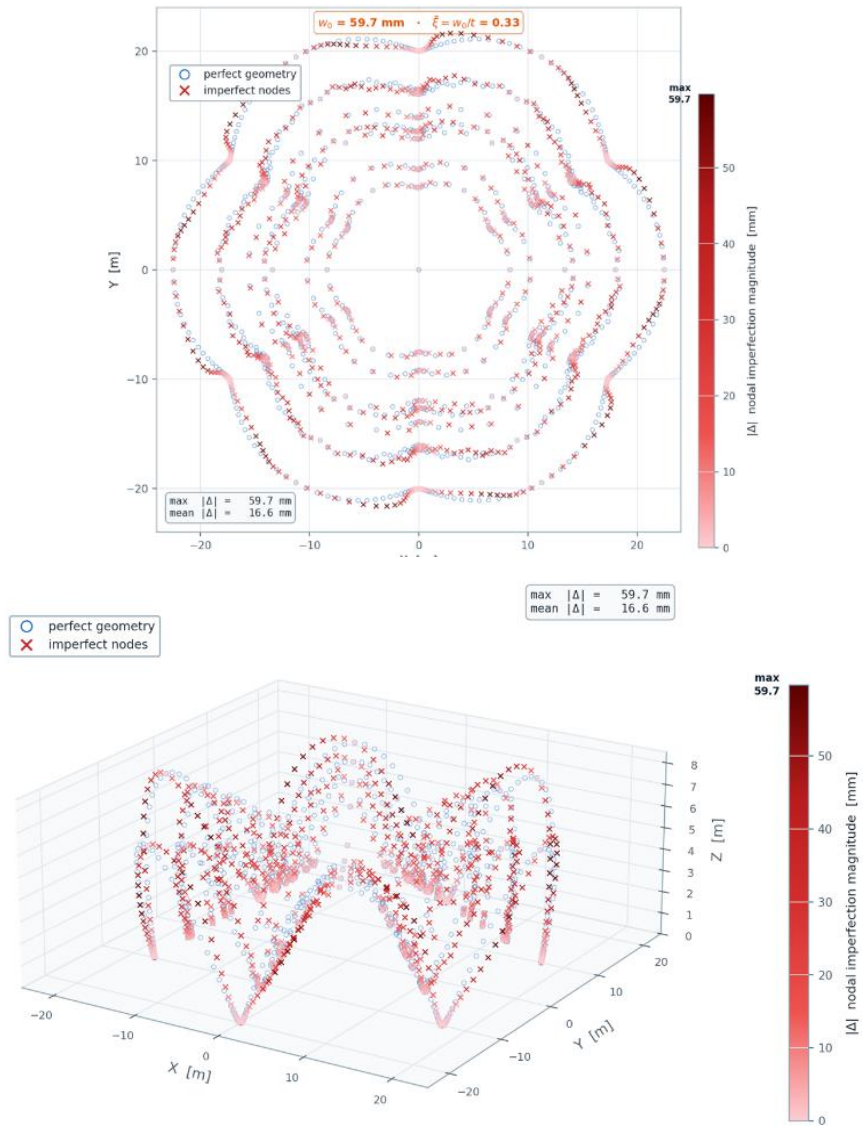


Fig. 18. Perfect and imperfect nodal geometries of the PH-06 model for $w_0 = 59.7$ mm ($\xi = 0.33$, dimensionless): (a) plan view, (b) 3D view; color: $|\Delta|$ (mm)

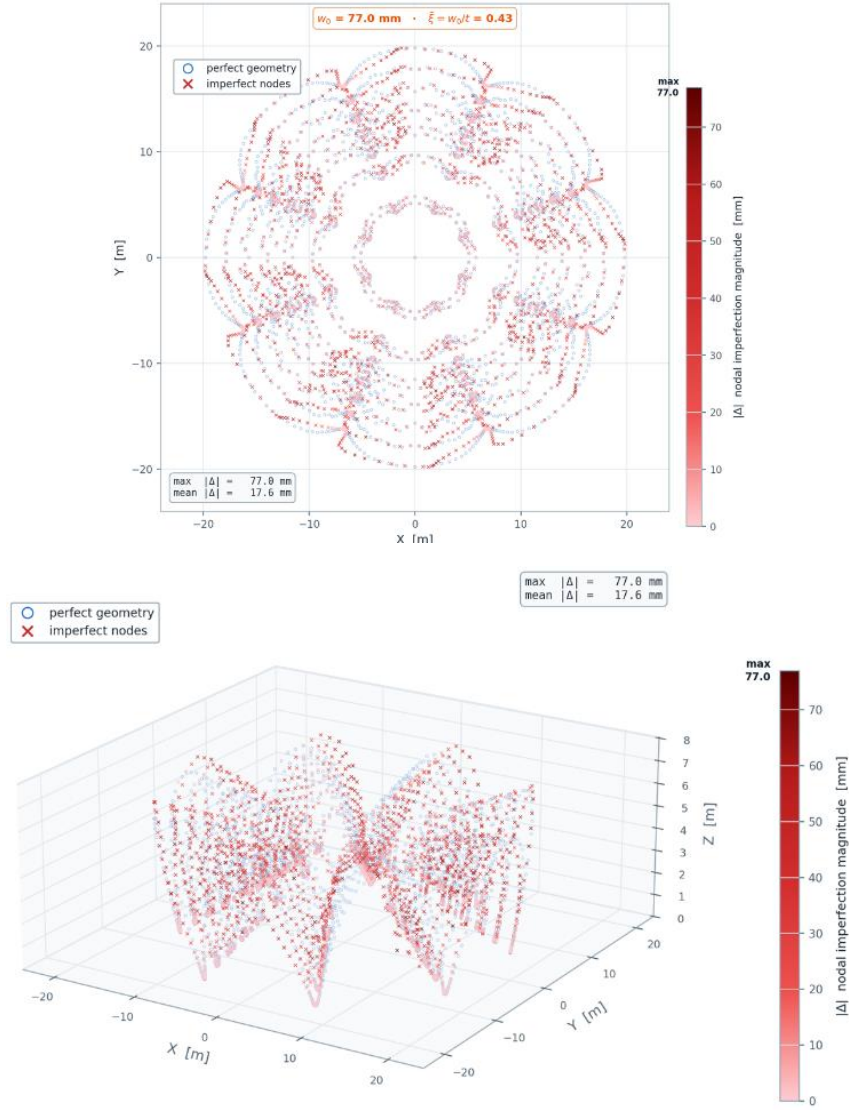


Fig. 19. Perfect and imperfect nodal geometries of the PH-08 model for $w_0 = 77.0$ mm ($\xi = 0.43$, dimensionless): (a) plan view, (b) 3D view; color: $|\Delta|$ (mm)

5.1.2. Nonlinear Buckling Analysis

Due to their slenderness, shell-type structures are prone to buckling problems, making it fundamentally important to conduct nonlinear buckling studies incorporating large-deformation theory [20]. It is well known that, for complex structures, closed-form solutions to analytically determine critical buckling loads or their associated configurations do not currently exist. Therefore, finite element simulation approaches are employed. In this study, these simulations are carried out following formulations such as:

$$K_{total}(u)\Delta u = F_{ext} - F_{int}(u) \quad (56)$$

The total nonlinear system stiffness is given by:

$$K_{total}(u) = K_L - K_G(u) + K_{NL}(u) \quad (57)$$

Where K_L represents linear stiffness, K_G the geometric stiffness due to initial stresses, K_{NL} the material nonlinear stiffness, u the displacements and F the internal and external forces. Using SAP2000, this equation is solved iteratively by employing methods such as Newton-Raphson, analyzed models produced the following results:

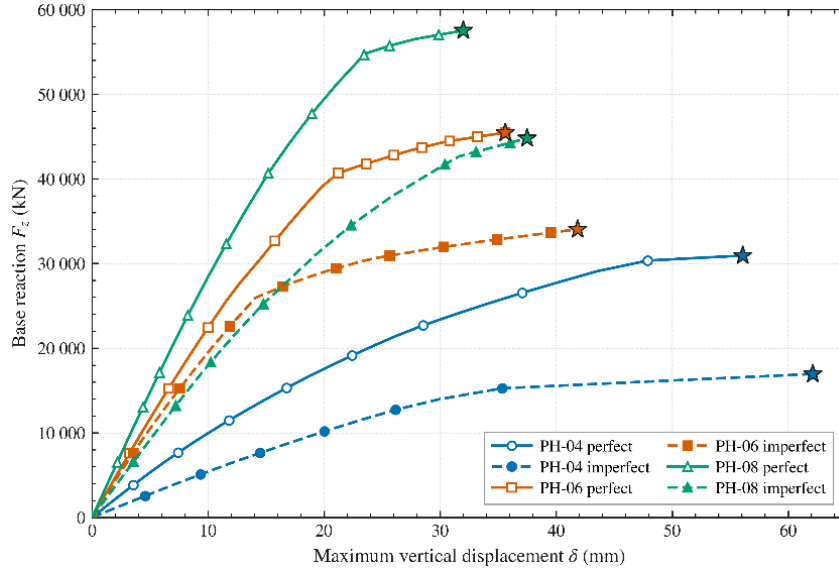


Fig. 20. Buckling in analyzed models under critical load

5.1.3. Knock-Down Factors

The imperfection sensitivity of each shell is quantified by the knock-down factor $\rho = F_{u,imp} / F_{u,per}$, obtained from the curves of Fig. 20. The resulting values, 0.548, 0.638 and 0.779 for PH-04, PH-06 and PH-08 (Fig. 21), lie well above the values of about 0.2 reported for cylindrical and spherical shells [23,24], confirming the lower sensitivity of anticlastic geometries [5]. Since PH-04 and PH-08 share nearly the same amplitude ($\xi = 0.45$ and 0.43), the higher ρ of PH-08 is geometric: shorter spans between supports confine the critical mode. These factors are adopted in this study as the stability reduction for each shell.

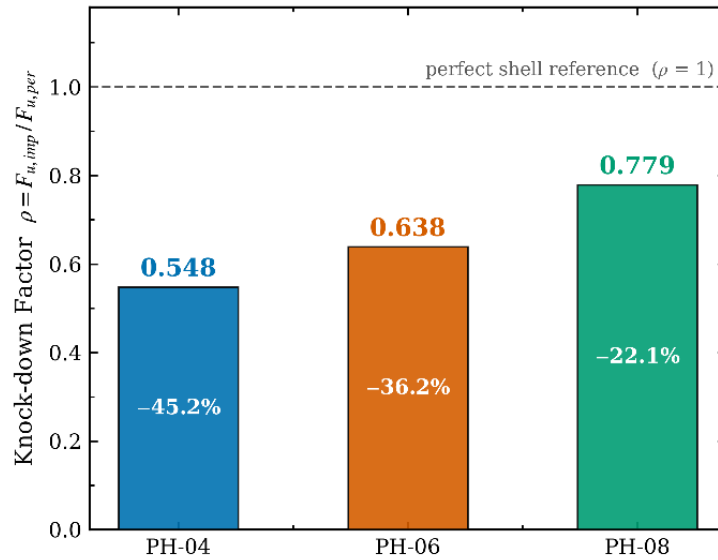


Fig. 21. Imperfection sensitivity on the shells

5.2. Modal Analysis and Vibration Characteristics

Given that the analysis is nonlinear and conducted in the time domain under accidental seismic excitations, it is essential to understand the vibration characteristics of shell-type structures. This understanding is achieved by solving the modal problem in terms of eigenvectors [33]:

$$\det[K - \Omega^2 M] = 0 \quad (58)$$

Where K represents the stiffness matrix, $\mathcal{D}$ is the diagonal matrix of eigenvalues, and M is the mass matrix. The modal shapes and fundamental periods of the studied models are defined accordingly:

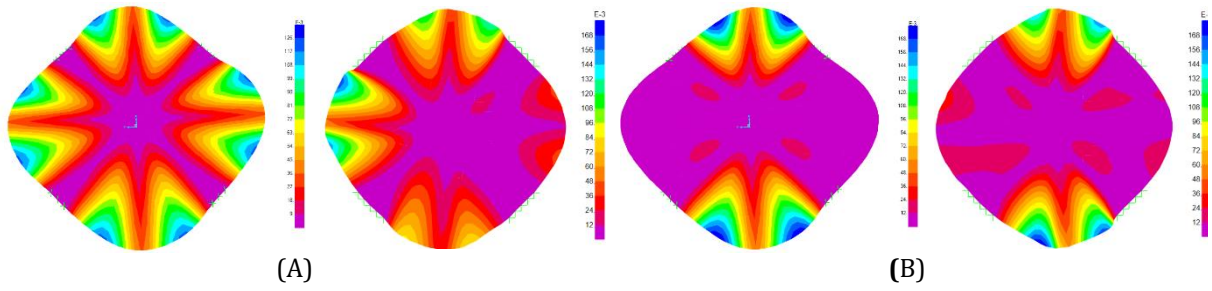


Fig. 22. Comparative modal shapes of the PH-04 model in mode 1(A) and 2 (B) (perfect geometry left, imperfect geometry right)

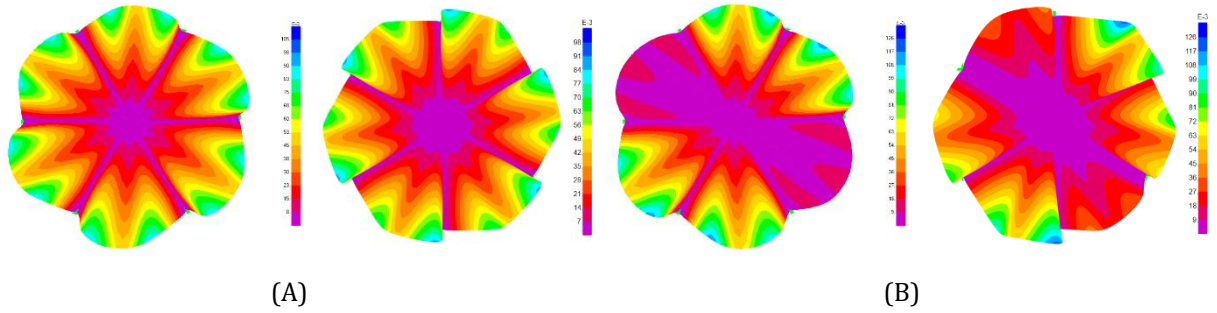


Fig. 23. Comparative modal shapes of the PH-06 model in mode 1(A) and 2 (B) (perfect geometry left, imperfect geometry right)

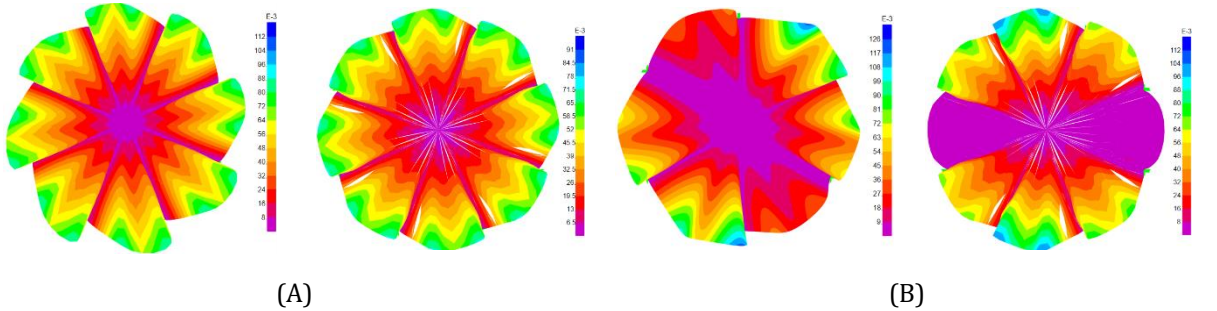


Fig. 24. Comparative modal shapes of the PH-08 model in mode 1(A) and 2 (B) (perfect geometry left, imperfect geometry right)

Table 4. Fundamental periods of the three shell models, perfect and imperfect geometry

MODE	Model PH-04		Model PH-06		Model PH-08	
	perfect	imperfect	perfect	imperfect	perfect	imperfect
1	0.59 sec	0.67 sec	0.49 sec	0.56 sec	0.26 sec	0.31 sec
2	0.56 sec	0.63 sec	0.46 sec	0.51 sec	0.25 sec	0.29 sec
3	0.56 sec	0.62 sec	0.46 sec	0.49 sec	0.25 sec	0.28 sec

The modal analysis of anticlastic shells is complex. Their fundamental vibration modes are not purely translational or rotational, due to their intrinsic geometric stiffness derived from the double curvature. This characteristic directly influences their modal configurations, leading to combined stress states involving bending moments, tension, and compression distributed throughout the entire surface. Determining these modal parameters is crucial because they provide insight into the dynamic behavior of the structure, including the distribution of deformations, resonance frequencies, and potential vulnerability under transient or extreme loading conditions.

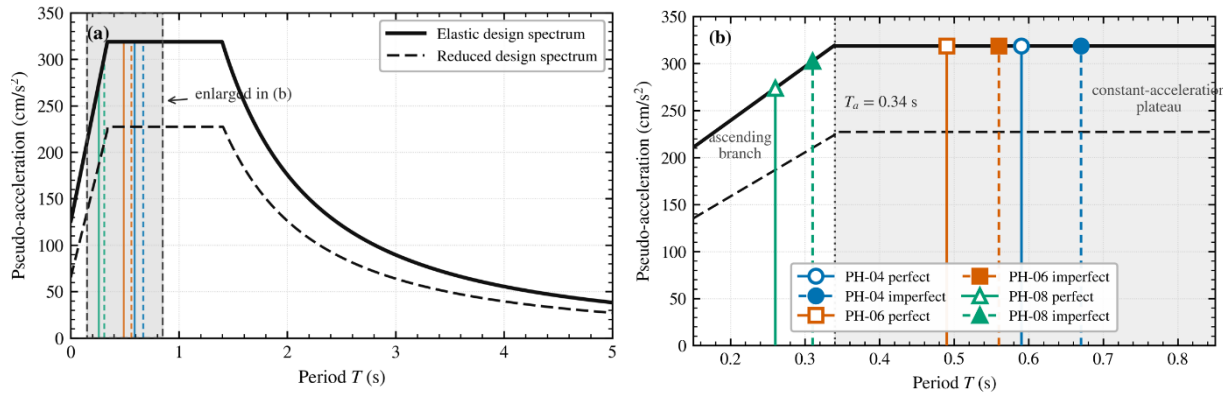


Fig. 25. Fundamental periods of the three shell models, in their perfect and imperfect configurations, placed on the design spectrum

For shell structures, which typically exhibit complex curvature and coupling between bending and membrane actions, accurate identification of the fundamental modes ensures that nonlinear time-history analyses under seismic events correctly capture the interplay between material nonlinearity, geometric effects, and dynamic response.

5.3. Stress State Under Design Load Combinations

The resultant stresses in the analyzed models incorporate the previously defined load combinations and are formulated within the framework of finite strain theory, leading to the following outcomes:

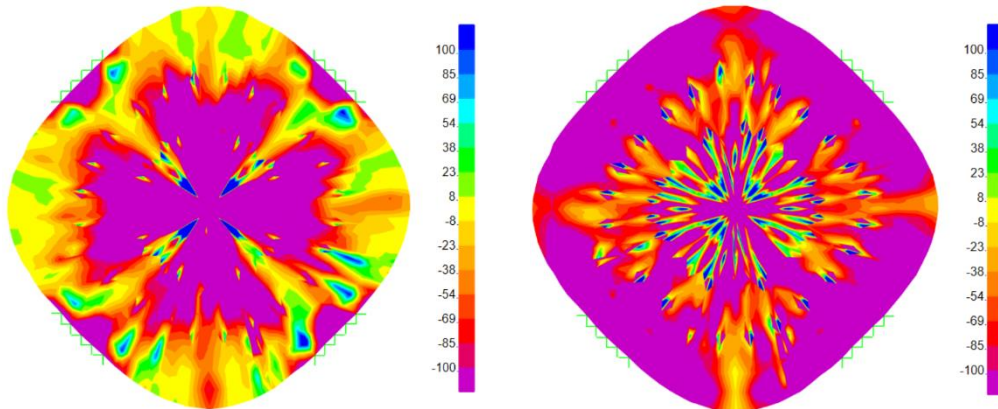


Fig. 26. Resultant forces of the PH-04 Model longitudinal (left) transversal (right) in N/mm²

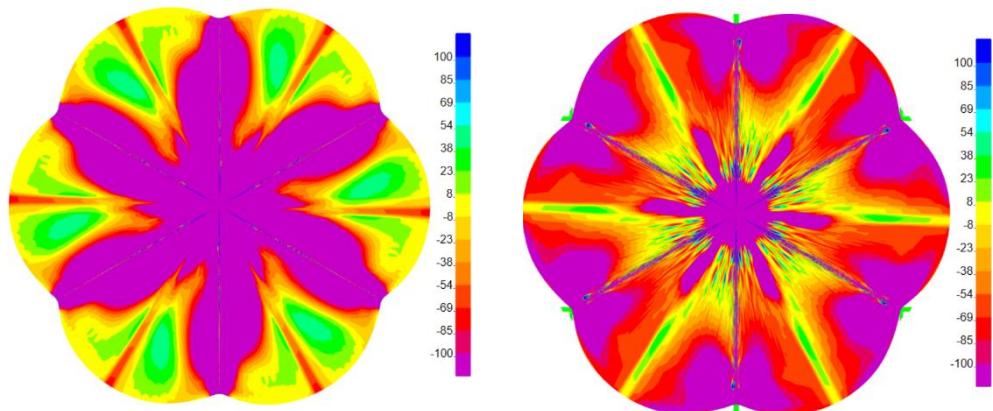


Fig. 27. Resultant forces of the PH-06 Model longitudinal (left) transversal (right) in N/mm²

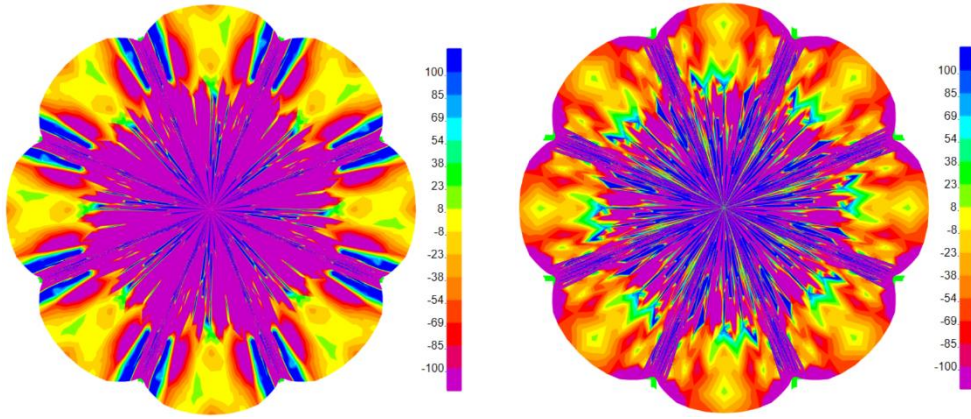


Fig. 28. Resultant forces of the PH-08 Model longitudinal (left) transversal (right) in N/mm^2

5.4. Reinforcement Design

The design of the shell structures was carried out in accordance with the Eurocode provisions previously described in Chapter 4 of this study. The numerical analysis is conducted in three fundamental stages after the design forces have been determined. The first stage involves verifying the cracking zones and intensity levels in the cross section of the concrete shells. Subsequently, the required reinforcement ratios are obtained for both the upper and lower layers in both directions. Finally, the reinforcement design using steel bars is established, which provides the basis for transparent nonlinear time-history analyses and subsequent verification using the multi-layered method.

5.4.1. Cracking Assessment

The cracking zones and intensity levels in the study models are shown, using a color scale as a reference. Blue indicates areas where the shell remains uncracked, while red represents regions experiencing the highest cracking intensity.

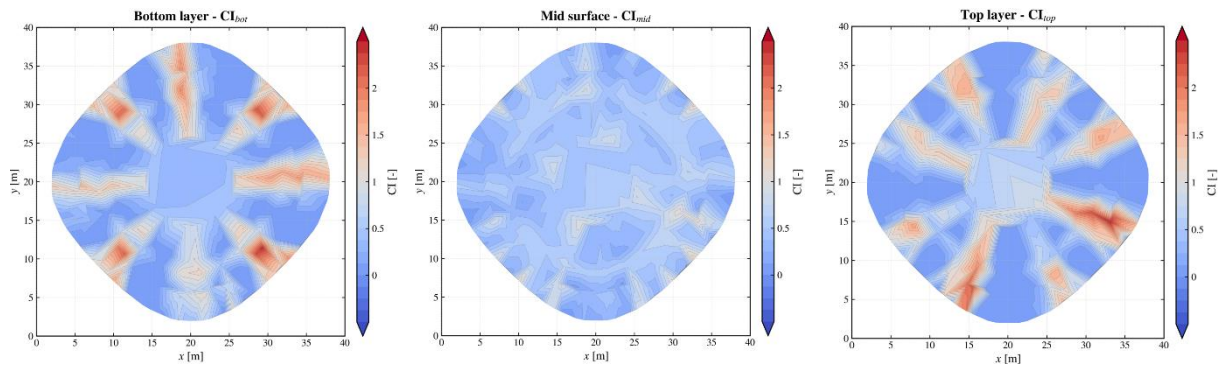


Fig. 29. Cracking of the PH-04 model (if $CI > 0 = \text{cracked zone}$)

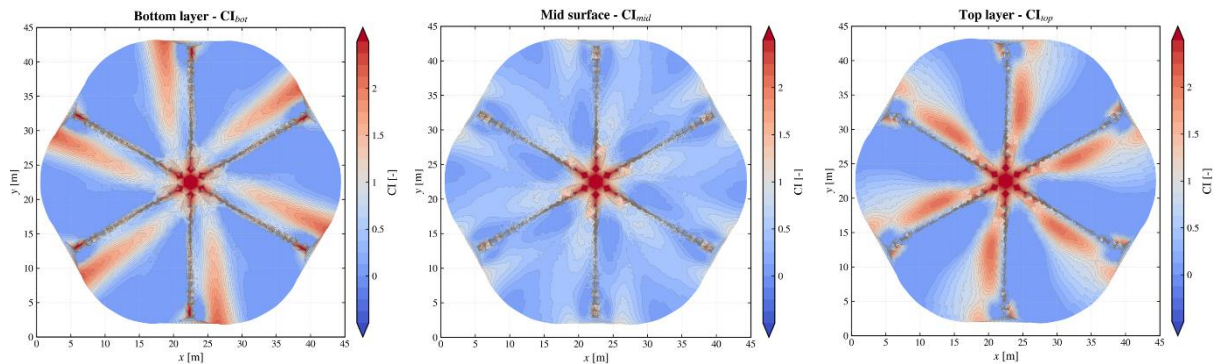


Fig. 30. Cracking of the PH-06 model (if $CI > 0 = \text{cracked zone}$)

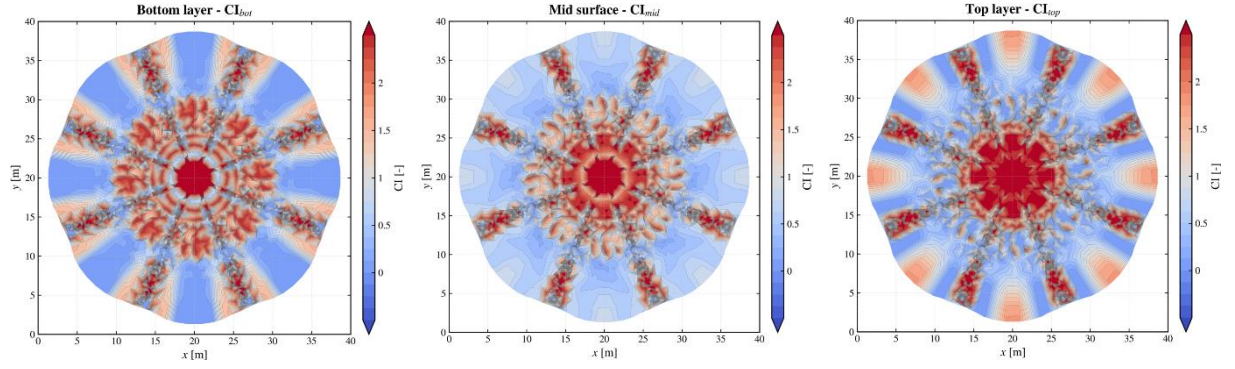


Fig. 31. Cracking of the PH-08 model (if $CI > 0 = \text{cracked zone}$)

The cracking maps of Figs. 29 to 31 must be interpreted in the light of the assumption stated in Section 4.2: the criterion of Eq. (38) evaluates concrete cracking only and does not assess crushing. The results are therefore conservative with respect to cracking every region in which the criterion is exceeded is reported as cracked but non-conservative with respect to the compressive regime, since the loss of capacity that follows the peak compressive strain is not detected. Under the bilateral loading and unloading cycles imposed by the seismic records, this asymmetry matters most where the compressive demand is highest, namely at the supports and at the peak points of the surface, and it implies that the ductility available in those regions is overestimated. The extent of this limitation is discussed in Section 6.

5.4.2. Required Reinforcement Area

Under the hybrid framework of Section 4, the ratio between the required and the minimum steel area, $A_{s,req}/A_{s,min}$, is mapped in Fig. 32 for the governing bottom layer of each model:

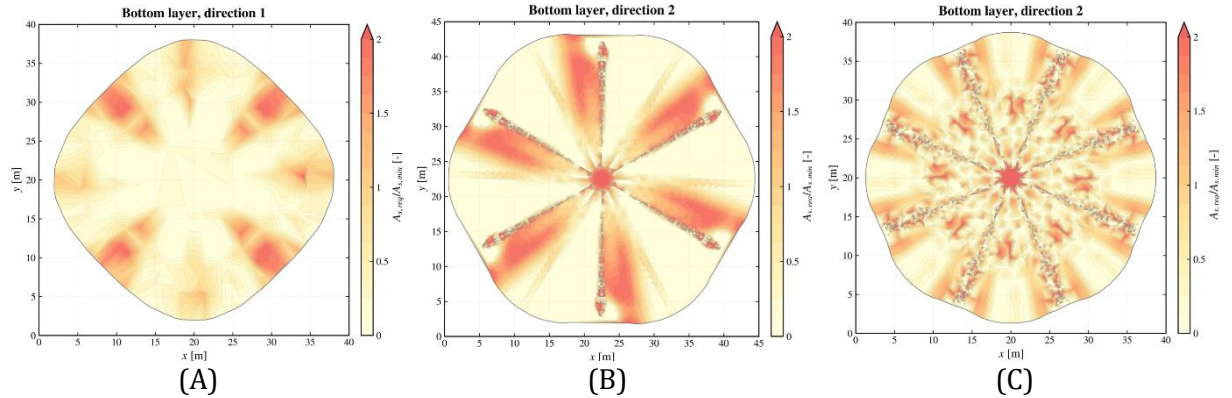


Fig. 32. Maximum $A_{s,req}/A_{s,min}$ ratio (dimensionless) in the governing bottom layer: (A) PH-04, direction 1; (B) PH-06 and (C) PH-08, direction 2

The reinforcement design, conducted under a hybrid framework (NTC-2023 actions and minimum steel; Eurocode 2 with Wood–Armer correction [34] for the conversion of shell forces into steel areas), demonstrates that form-finding by the force density method governs the reinforcement demand. The code minimum [$A_{s,min} = 337 \text{ mm}^2/\text{m}$] [35] controls 74.0%, 49.7% and 59.1% of the surface of PH-04, PH-06 and PH-08, respectively. This provides quantitative evidence that the optimized geometry preserves a near-membrane stress state under seismic combinations. The residual demand is not randomly distributed: it concentrates systematically in the bottom layer, along the meridian bands and at the crown, the same regions identified as critical in the cracking analysis. A uniform two-layer mesh of #4 bars at 125 mm (317.8 n/m^2), supplemented locally along the meridian bands, satisfies 100%, 98.2% and 99.1% of the demand. The residual peaks are confined to mesh singularities and supports. Because the multilayered formulation explicitly accounts for the compressive contribution of the reinforcement and the actual lever arm, it requires [6 - 12%] less steel than the Eurocode 2 sandwich model. This result shows that the sandwich idealization is systematically conservative for seismically loaded thin shells.

Table 5. Max required steel area and adopted two-layer reinforcement of the three shell models

LAYER	Model PH-04		Model PH-06		Model PH-08	
	As,req,max (mm ² /m)	reinforcement	As,req,max (mm ² /m)	reinforcement	As,req,max (mm ² /m)	reinforcement
Top, direction x	260	#4 @ 125 mm	1 069	#4 @ 125 mm	2 442	#4 @ 125 mm
Top, direction y	354	#4 @ 125 mm	1 666	#4 @ 125 mm	2 141	#4 @ 125 mm
Bottom, direction x	1 055	#4 @ 125 mm	3 929	#4 @ 125 mm	4 022	#4 @ 125 mm
Bottom, direction y	686	#4 @ 125 mm	4 326	#4 @ 125 mm + local*	4 424	#4 @ 125 mm + local*
Area governed by As,min (%)	74.0		49.7		59.1	
Area covered by reinforcement (%)	100		98.2		99.1	

5.5. Nonlinear Seismic Time-History Analysis

This type of analysis is commonly employed to evaluate the nonlinear dynamic response of structures under seismic loading in an incremental manner, by solving the equations of motion as follows:

$$Kx(t) + C\dot{x}(t) + M\ddot{x}(t) = f(t) \quad (59)$$

Where K is the stiffness matrix, C is the damping matrix, M represents the diagonal mass matrix, and x , $\dot{x}$, and $\ddot{x}$ are the displacement, velocity, and acceleration vectors of the structure, respectively, with f denoting the vector of applied forces.

The seismic signals utilized in this study were synthetically generated in accordance with the provisions of Mexican seismic regulations, representing a stiff soil site condition in Mexico City, and are defined as follows:

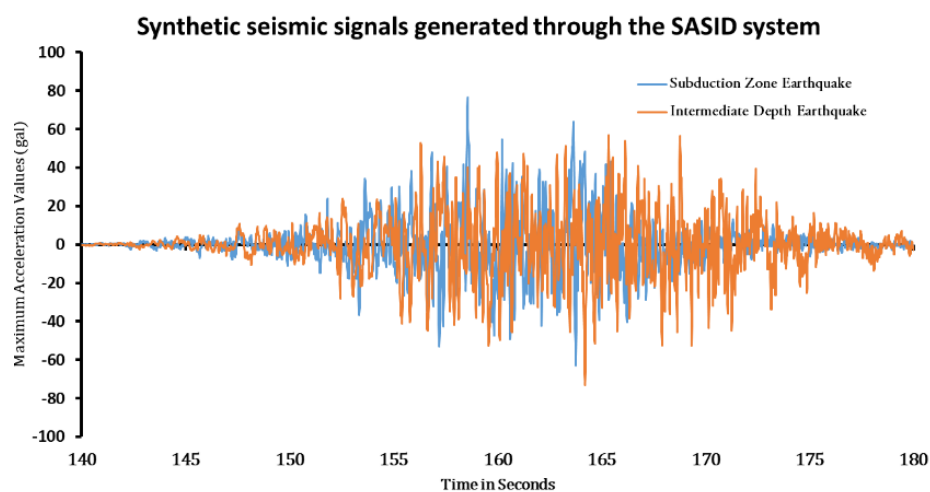


Fig. 33. Synthetic Seismic Records used in Time-History Analysis

The nonlinear time-history analyses were not started from an unstressed configuration. A nonlinear static case comprising dead load and service live load was solved first, and the seismic cases were defined to continue from its final state, so that the stresses and deformations already present under operating conditions are inherited as the initial condition of the seismic response

rather than superimposed afterwards. Consistently, the mass source adopted for both the modal and the time-history analyses comprised 100% of the dead load and 100% of the instantaneous live load $W_a = 0.69 \text{ kN/m}^2$, prescribed for roofs in Table 6.1.2.2 of the Mexican Complementary Technical Standards on Criteria and Actions for Structural Design, whose Section 6.1.2.2 requires the instantaneous live load to be adopted for accidental actions such as earthquake [8]. Time-dependent material effects were not included in this initial state: creep and shrinkage are excluded from the model, and their omission is discussed in Section 6.

5.5.1. Hysteretic Model Used in The Analysis

A hysteretic model based on the formulation proposed by Takeda et al. (1970) [36] is employed to simulate the nonlinear behavior of reinforced concrete structures and of elements with degradable stiffness. This model allows reproducing the bilinear post-yield hardening and the progressive degradation of material stiffness as a function of cyclic displacements.

The Takeda hysteretic model effectively reproduces the experimentally observed degradation of secant stiffness in reinforced concrete under cyclic loading. Its formulation employs a modified unloading rule toward the origin, governed by secant stiffness and the amplitude of prior maximum displacements, thus capturing stiffness degradation and the pinching effect typical of cracking materials. Consequently, its application in the nonlinear analyses of this study is justified, as it allows realistic simulation of damage progression and seismic response in concrete shells, supporting the evaluation and validation of geometrically optimized structural models.

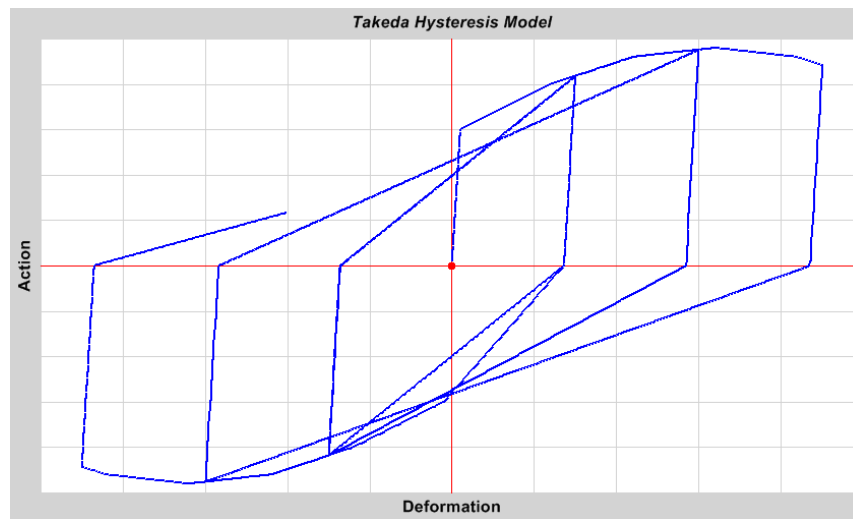


Fig. 34. Takeda's Hysteretic Model [36]

5.5.2. Nonlinear Multi-Layered Model

The implementation of a multi-layer model in nonlinear time-history analyses aims to ensure compatibility with the nonlinear material models, large-deformation theory, and hysteretic behavior models, in order to accurately identify the damage zones and intensity levels experienced by shell-type structures under seismic loading. The multi-layer approach enables a detailed representation of the stress and strain distribution through the shell thickness, considering a linear variation of deformations across the depth. This improves accuracy compared to the three-layer sandwich model [37,38], which assumes a uniform distribution.

The main rationale for employing a multilayered model in the nonlinear time-history analyses conducted in this research lies in the need to evaluate the hypotheses and assumptions prescribed in the Eurocode, specifically in Annex LL, where it is assumed that the reinforcement in concrete shell structures only resists tensile stresses, while the central concrete layer of the three-layered element is responsible for carrying compressive stresses. Unlike the simplified sandwich model, the multilayered approach enables an accurate representation of compressive reinforcement behavior, which the former cannot adequately capture. This limitation often leads to over-dimensioned and excessively heavy concrete shells.

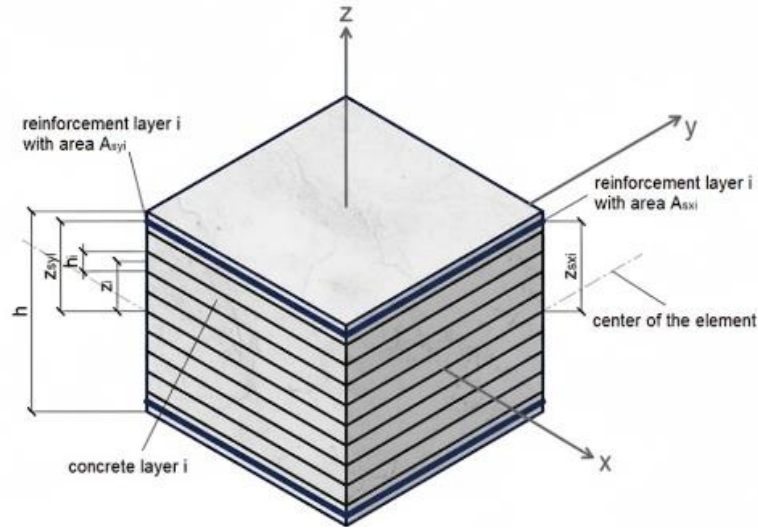


Fig. 35. Reinforced concrete shell element idealized by Kollegger et al. [37]

Therefore, the employment of advanced nonlinear modeling techniques, such as the multilayered model, allows for a more accurate and reliable structural analysis, fostering the safe slenderness optimization of shell-type elements, which represents one of the core design principles of this investigation.

The compressive contribution of the reinforcement reported here does not imply that compression reinforcement is required as such in a thin shell. It follows from the reversal of the stress state under bilateral seismic excitation: a fiber that carries tension in one half-cycle carries compression in the next, so that reinforcement placed to resist the tensile demand is subsequently compressed [30,39]. What the multilayered analyses show is that this contribution is not negligible: at the sections of highest demand, the reinforcement layers carry a measurable share of the peak compressive force, which the three-layer model assigns entirely to the concrete core. The three-layer sandwich model of Annex LL assigns that force entirely to the concrete core, and the consequence is systematic: the concrete layer is sized for a demand it does not carry alone, and the resulting section is larger than the response requires.

The principal equation of the multilayer method is defined as:

$$K_{multilayer}(u)\Delta u = F_{ext} - F_{int}(u) \quad (60)$$

$$K_{multilayer} = \sum_{i=1}^n \int_A B_i^T C_i B_i dA \quad (61)$$

Where B represents the strain matrix for each layer, C the stiffness matrix for each layer, and A the finite element of the shell-type surface.

Only the main simplified formulations of the multilayered method are presented herein; readers are encouraged to consult [40] for a comprehensive understanding.

5.5.3. Nonlinear Time-History Results in Models

The three shells were subjected to the synthetic records EQ ST 01 and EQ SM 02 ($\Delta t = 0.02$ s, 45 s), applied to the gravity-deformed state of the imperfect geometry. The lateral displacement of three control joints and the base shear V_x were recorded for each model. The plots show the total displacement u_x with the initial state u_0 as reference, whereas the reported quantities are excursions from u_0 , that is, the displacement component induced by the earthquake. The response was evaluated through the peak displacement, which quantifies the instantaneous demand; the residual displacement at the end of the record, which quantifies the irreversible deformation; and the linearity of the $V_x - u_x$ relationship, expressed by the correlation coefficient R .

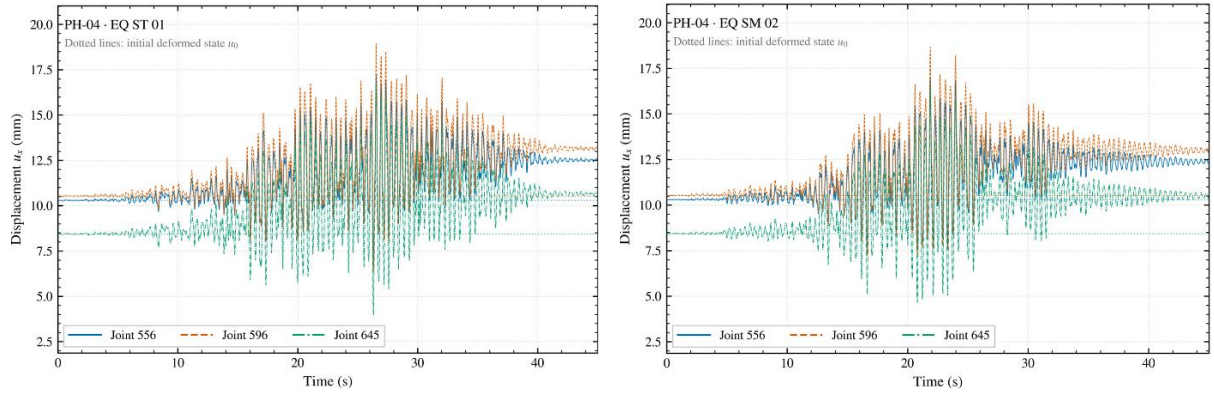


Fig. 36. Maximum lateral displacements in PH-04

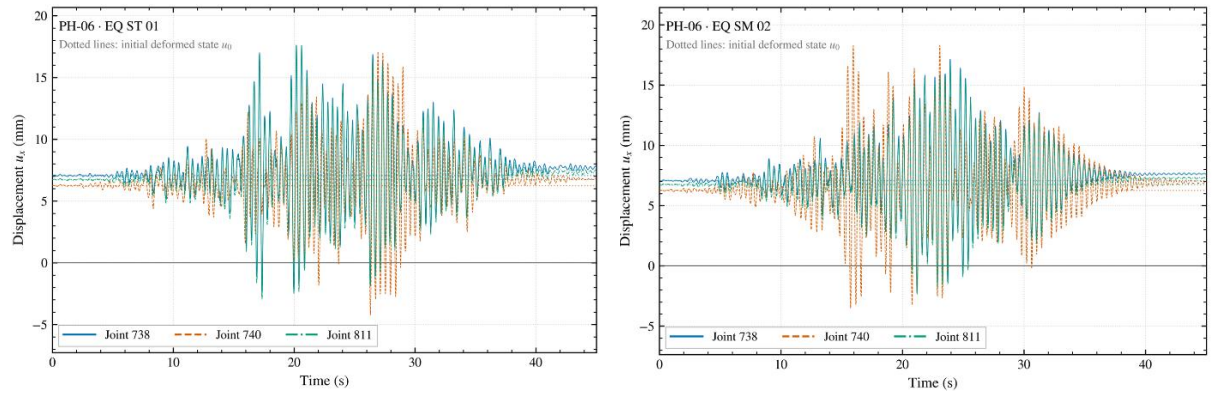


Fig. 37. Maximum lateral displacements in PH-06

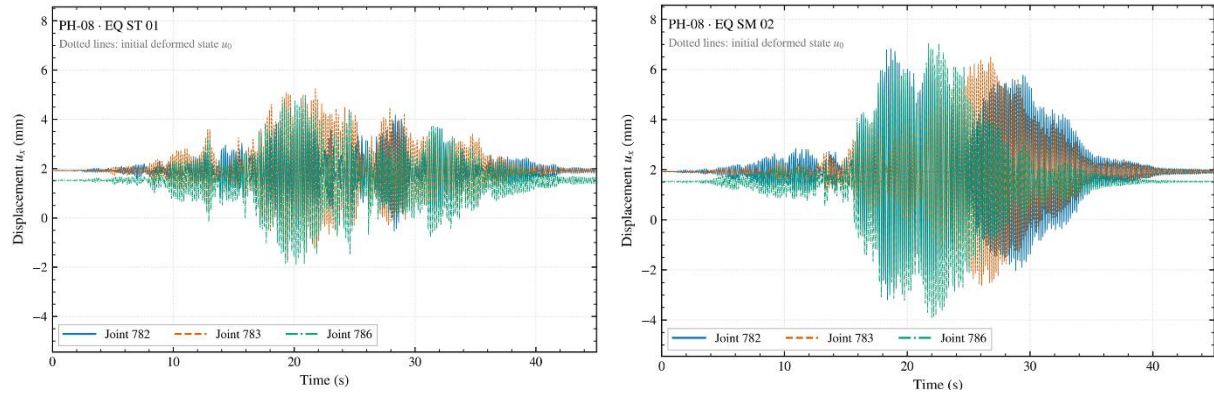


Fig. 38. Maximum lateral displacements in PH-08

Figs. 36 to 38 show the displacement histories of the three control joints of each shell under both records. The joints of each model respond consistently with one another and between records. In PH-04, the three joints end the record displaced above u_0 , whereas in PH-06 and PH-08 they return practically to their initial position.

The peak excursions were 8.2 to 8.4 mm in PH-04, 10.9 to 12.0 mm in PH-06, and 3.5 to 5.5 mm in PH-08, with no monotonic trend with the number of supports. PH-06 reached the largest peak despite a lower base shear than PH-08 because of its lower lateral stiffness. The residual displacement, in contrast, ranked the models monotonically: 2.4 to 2.6 mm in PH-04 (29% to 31% of the peak), 0.6 to 0.8 mm in PH-06 (5% to 6%), and 0.06 mm or less in PH-08 (below 2%). Because it does not depend on the instantaneous amplitude, this is the indicator that reflects accumulated damage. To compare the models on a common scale, Fig. 39 superimposes the response of the homologous joints 596, 740, and 786. Figs. 40 to 42 present the relationship between base shear and displacement at the control joints of each shell.

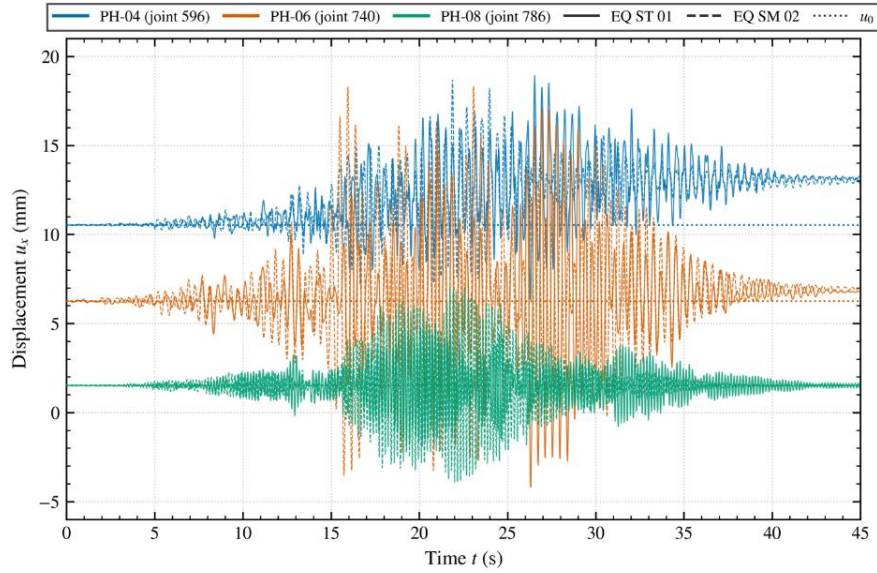


Fig. 39. Lateral displacement histories of the homologous control joints of PH-04 (596), PH-06 (740), and PH-08 (786) under both records

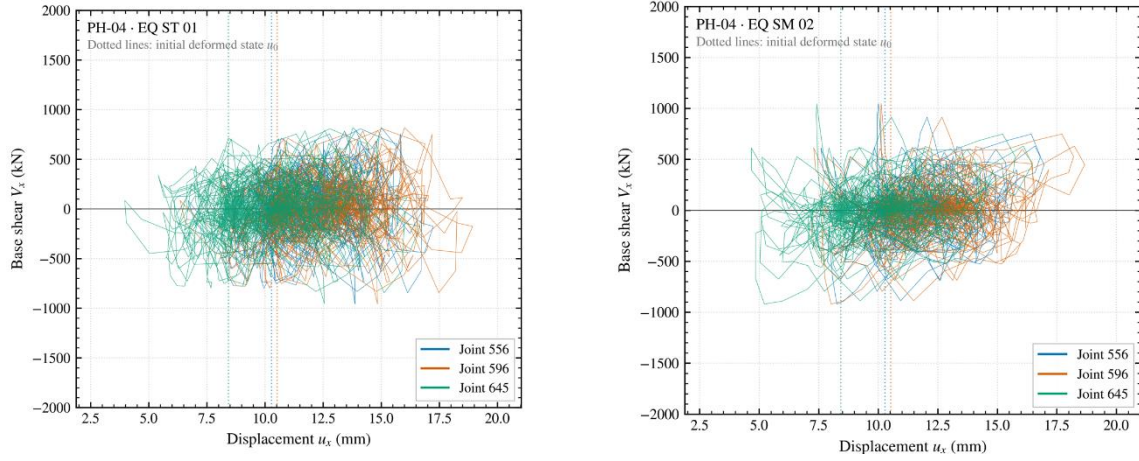


Fig. 40. Base shear V_x versus lateral displacement u_x of the control joints of PH-04 under EQ ST 01 (left) and EQ SM 02 (right)

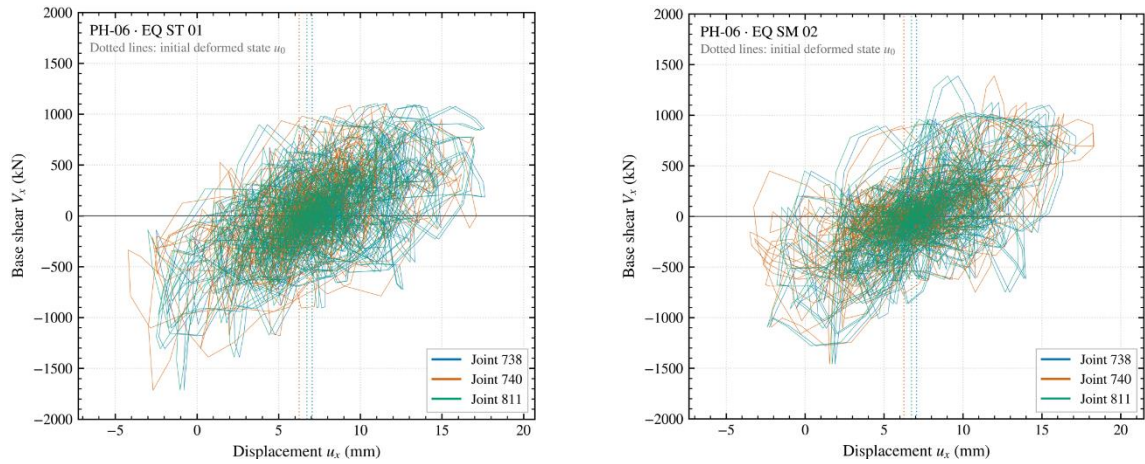


Fig. 41. Base shear V_x versus lateral displacement u_x of the control joints of PH-06 under EQ ST 01 (left) and EQ SM 02 (right)

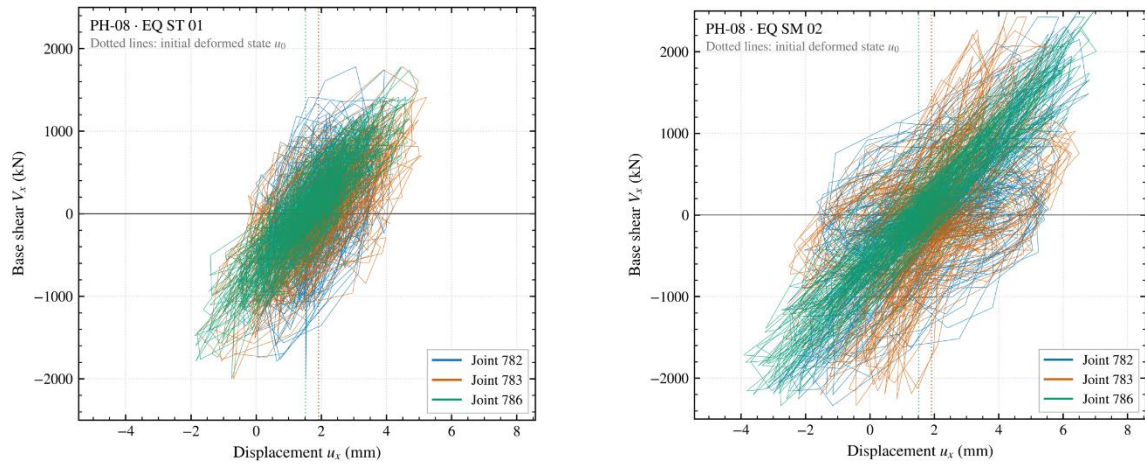


Fig. 42. Base shear V_x versus lateral displacement u_x of the control joints of PH-08 under EQ ST 01 (left) and EQ SM 02 (right)

The shape of the response changes with the number of supports. In PH-08 it is a narrow band aligned with a straight line ($R = 0.88$ to 0.96); in PH-06, an inclined and more scattered cloud ($R = 0.55$ to 0.59); and in PH-04, a practically horizontal cloud, shifted toward positive values and with no correlation between force and displacement ($R = 0.13$ to 0.20). Fig. 43 combines the loops of the homologous joints with the stiffness obtained by linear regression, which allows the mechanism to be quantified.

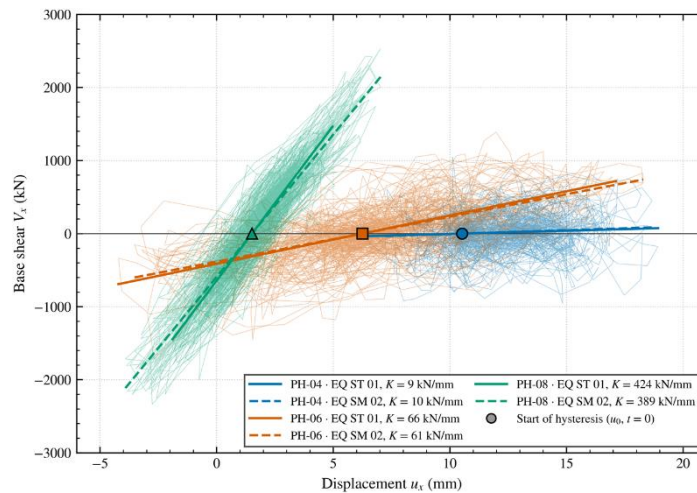


Fig. 43. Base shear–displacement loops of the homologous control joints of the three shells, with the stiffness K obtained by linear regression; markers indicate the start of each loop ($u_0, t = 0$)

5.6. Design Efficiency of The Multilayer Formulation Relative to The Three-Layer Sandwich Model

The three shells were designed with both procedures under the design load combination. The three-layer model (EN 1992-2, Annex LL) is a local, lower-bound check based on a single linear elastic stress field: the reinforcement resists tension only, and the most highly stressed element governs the thickness of the shell. The multilayer model integrates nonlinear concrete and steel layers through the thickness, allowing the reinforcement to act in both tension and compression, and determines thickness from the global capacity, with no reinforcement yielding under the design load, thereby accounting for load redistribution.

For $h = 180$ mm, the multilayer model allows 2.3, 1.3, and 3.7 times the live load permitted by the three-layer check for PH-04, PH-06, and PH-08, respectively. The three-layer thickness increases more rapidly with w_L , largely because of the elastic bending peaks at the supports, which add up to 70 mm in PH-06 (Fig. 44) and are redistributed by cracking in the nonlinear analysis.

At $w_L = 6 \text{ kN/m}^2$, the multilayer design reduces the concrete volume and self-weight by 33%, 26%, and 40% (Figs. 45 and 46), and by 24%, 7%, and 38% when the support elements are excluded; the PH-08 value is a lower bound because its thickness remains at the 150 mm minimum.

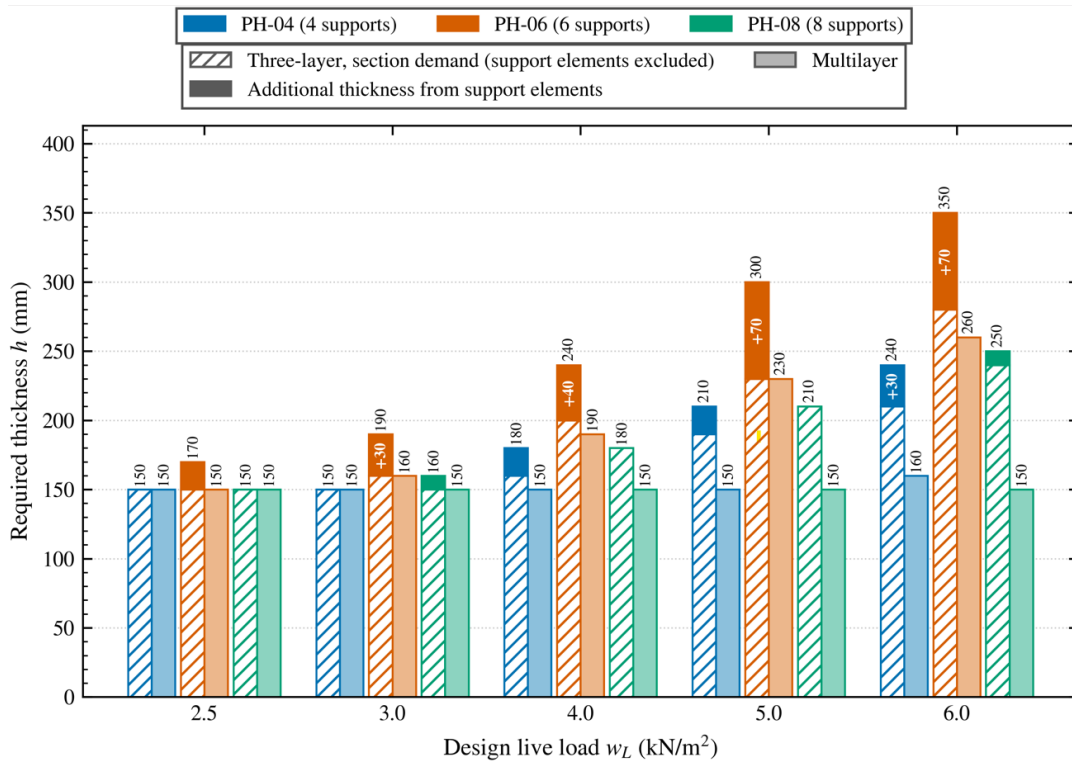


Fig. 44. Required thickness versus design live load, hatched three-layer without supports, light multilayer and solid additional thickness for support elements

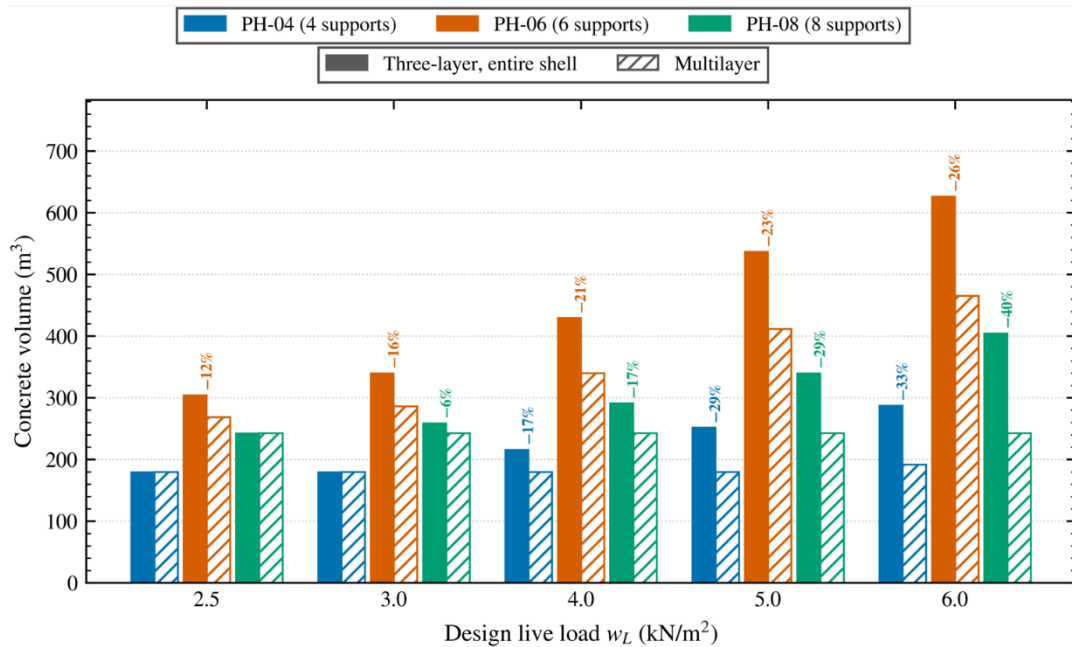


Fig. 45. Concrete volume versus design live load, solid: three-layer; hatched: multilayer

Fig. 45. compares the concrete volume required by the two design procedures for the three shells over the range of design live loads considered. The corresponding self-weight, which governs both the permanent gravity load and the seismic mass of the shell, is shown in Fig. 46.

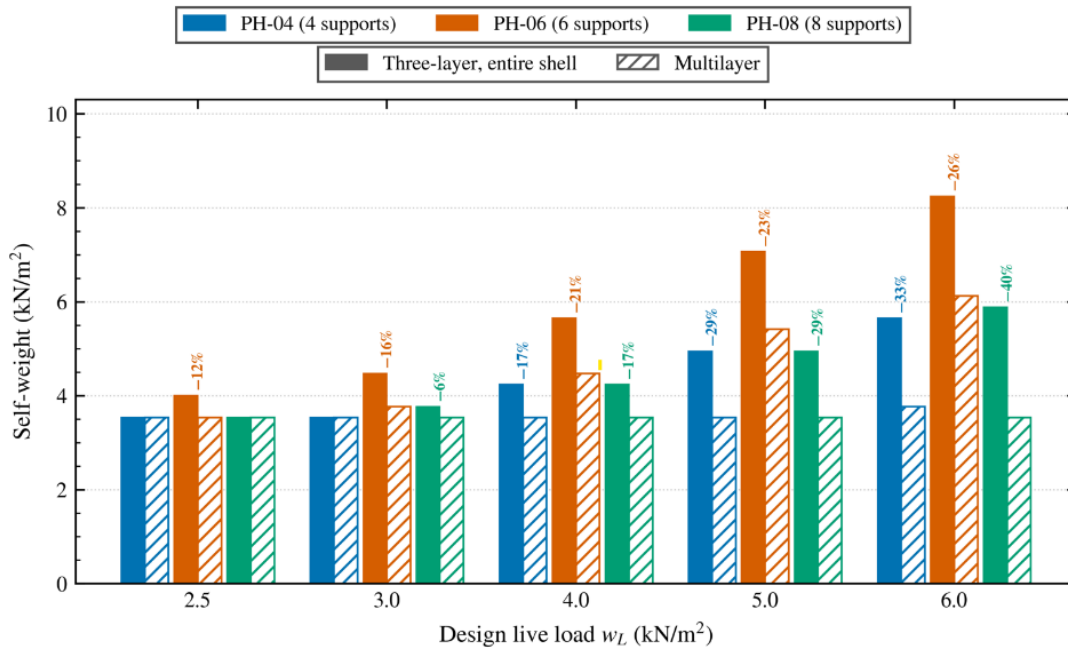


Fig. 46. Concrete self-weight versus design live load, solid: three-layer; hatched: multilayer

6. Limitations and Future Research

The methodology presented provides a robust assessment of the seismic performance and nonlinear behavior of concrete shell roofs, but its analytical scope must be delimited. Declaring these computational and phenomenological boundaries is necessary both for the correct interpretation of the results and for establishing the framework within which the findings apply. The main limitations are the following.

- Concrete crushing. The constitutive model adopted captures tensile cracking and global buckling instability but does not explicitly simulate concrete crushing. Localized concentrations of compressive stress, which may precipitate brittle failure in regions of high curvature and in the vicinity of the supports, are therefore not evaluated. The available ductility is consequently overestimated, and the overestimation is not uniform across the models: it is most significant in the eight-support shell, which is the stiffest configuration and the one that develops the least ductility.
- Exclusion of soil–structure interaction. The model assumes an infinitely rigid base [43]. Neglecting foundation flexibility and the kinematic and inertial interaction between soil and structure prevents any assessment of how radiation damping and the lengthening of the fundamental period would modify the stress distribution in the shell. This is particularly relevant for the Valley of Mexico, where the soft lacustrine deposits amplify long-period demand, whereas the fundamental periods obtained here lie in the short-period range.
- Time-dependent effects. Creep and shrinkage were not included in the model [44]. In thin-walled shells these phenomena induce progressive long-term deflections that redistribute the internal stress field and reduce the effective stiffness and may therefore aggravate the sensitivity of the structure to instability. The interaction between time-dependent behavior and seismic demand cannot be assessed from the analysis reported here.

Finally, the treatment of initial defects was restricted to purely geometric imperfections. In construction practice the limit performance of a shell is also governed by material deviations: the spatial variability of the cast thickness and the stochastic reduction of the initial modulus of elasticity are both relevant to a realistic assessment of risk.

Addressing these limitations defines the authors' research agenda. The development of a comprehensive probabilistic framework for shell structures, incorporating uncertainty in thickness and material stiffness, is the objective of the next phase of this work. In parallel,

subsequent studies will incorporate recorded ground motions, explicit modeling of the supporting half-space, and the assessment of the long-term rheological deterioration of the concrete.

7. Conclusions

This study showed that the geometry obtained with the force density method (FDM) constitutes the first line of seismic design for thin reinforced concrete shells. Three anticlastic shells with four, six, and eight supports were evaluated within a hybrid framework that combined the actions and minimum reinforcement of NTC-2023, the sectional design of Eurocode 2 with the Wood–Armer correction, and nonlinear time-history verification using a multilayer shell formulation.

Under synthetic ground motions for Mexico City, the three shells exhibited a controlled nonlinear response, dominated by compression and far from collapse. The response showed a clear dependence on the number of supports: from PH-04 to PH-08, the fundamental period decreased by 55.9%, the gravity deflection by 86%, and the residual displacement by 98%, so that stiffness increased and accumulated damage decreased monotonically. Only the peak seismic displacement departed from this trend, because PH-06 remained essentially elastic and developed the full seismic demand.

The geometric efficiency was reflected in the reinforcement: the minimum steel ($337 \text{ mm}^2/\text{m}$ per layer and direction) governed 74.0%, 49.7%, and 59.1% of the surface of PH-04, PH-06, and PH-08, respectively, indicating a predominantly membrane stress state that was satisfied by a uniform mesh of two layers of #4 bars at 125 mm, with local reinforcement along the meridian bands. In contrast, geometric imperfections affine to the first buckling mode reduced the ultimate capacity by 45.2%, 36.2%, and 22.1% and lengthened the fundamental periods by 13.6% to 19.2%. This sensitivity decreased with the number of supports, as the shorter spans confined the critical buckling mode, and remained below that reported for cylindrical and spherical shells.

This loss of capacity makes the margin recovered by a more rigorous analysis decisive. The three-layer model of Eurocode 2 proved conservative and uneconomical, although not unsafe: for a thickness of 180 mm, the multilayer model admitted 2.3, 1.3, and 3.7 times the live load allowed by the three-layer check, and at 6 kN/m^2 it reduced the concrete volume and self-weight by 33%, 26%, and 40%. The excess arises because the linear sectional criterion admits neither redistribution through cracking nor the actual internal lever arm of the layered section, which reduces the required reinforcement area by 10%. The multilayer representation is therefore a requirement, not a refinement, for the design of slender shells in seismic regions. The computed cracking zones coincided in location with the damage documented in Los Manantiales after the 2017 earthquake [41,42]: supports, valley lines, and free edges. Although this is a coincidence rather than a validation, it has scientific value: it indicates that, in shells whose efficiency derives from their geometry, the location of damage is governed by form rather than by the particular case, since the membrane state degrades in the same critical regions. This result is the starting point for a future study, because the models do not reproduce the actual geometry, thickness, or materials; its validation requires surveying the as-built structure, calibrating the model with ambient vibration measurements, and analyzing it with the 2017 records on lacustrine soil.

The main contribution is an integrated and reproducible framework that combines form-finding, affine imperfections, hybrid code-based design, and nonlinear multilayer verification, and that is transferable to other seismic regions by replacing the local codes and ground-motion records. Future work includes the multilayer comparison under seismic combinations and smaller thicknesses; a probabilistic framework based on FORM/SORM and Monte Carlo simulation; imperfections with multiple amplitudes, measured shapes, and variable thickness; and the inclusion of recorded ground motions, soil–structure interaction, creep, and shrinkage. Together, these developments will contribute to the safe and efficient reintroduction of thin concrete shells in seismic regions worldwide.

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